21-237: Math Studies Algebra I November 30, 2022

Lecture 34 : Some general ideas about modules

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1 Free modules

Recall: For a ring R and an R-module M, we say that M is **free** if M has a basis, i.e. a linearly independent generating set.

We can equivalently characterize $X \subseteq M$ as a basis for M if for every $m \in M$, there is a unique $(r_x)_{x\in X}$ such that $\{x : r_x \neq 0\}$ is finite and $m = \sum_{x\in X} r_x x$.

This sets up an isomorphism of R-modules between M and the module with elements $\{(r_x)_{x\in X} : r_x \in R, \{x : r_x \neq 0\}$ is finite$\}$ and pointwise operations, which we call $\text{Free}_R(X)$.

Fact: For any R-module N, there is a free module M such that N is isomorphic to a quotient of M.

Proof: Let $M = \text{Free}_R(N) = \{(r_n)_{n\in N} : \{n : r_n \neq 0\}$ is finite$\}$.

Let $\phi : M \to N$ via $\phi((r_n)_{n\in n}) = \sum_{n\in N} r_n n$. It's easy to see that ϕ is a surjective HM, so by the first isomorphism theorem, $N \simeq \frac{M}{\ker(\phi)}$. □

2 Direct sums and products of R-modules

Let R be a ring and let $(M_i)_{i\in I}$ be a family of R-modules, where I is a (potentially infinite) set.

The **(external) direct sum** of $(M_i)_{i\in I}$ is $\{(m_i)_{i\in I} : m_i \in M_i, \{i : m_i \neq 0\}$ finite$\}$, equipped with pointwise module operations.

The **direct product** of $(M_i)_{i\in I}$ is $\{(m_i)_{i\in I} : m_i \in M_i\}$ equipped with pointwise operations.

Note:

(1) The direct sum is a submoulde of a direct product.

(2) If I is finite, the direct sum is the direct product.

Trivial remark: For any set X, $\text{Free}_R(X)$ is the direct sum of an X-indexed family of copies of R.

Notation: $\bigoplus_{i\in I} M_i$ denotes the direct sum, $\prod_{i\in I} M_i$ denotes the direct product.

3 A categorical approach

Let $\mathcal{C}$ be a category, and let $(c_i)_{i\in I}$ be an I-indexed sequence of objects of $\mathcal{C}$.

A **product** of $(c_i)_{i\in I}$ is an object d together with an I-indexed family $(\pi_i : d \to c_i)_{i\in I}$ of morphisms such that for all objects d' and all families of morphisms $(\pi'_i : d \to c_i)_{i\in I}$, there is a unique $\alpha : d' \to d$ such that $\pi_i \circ \alpha = \pi'_i$, i.e. the following diagram commutes:

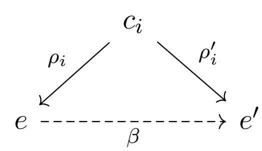
$$\begin{array}{ccc} d' & \overset{\alpha}{\dashrightarrow} & d \\ & {\scriptstyle \pi'_i}\searrow \quad \swarrow{\scriptstyle \pi_i} & \\ & c_i & \end{array}$$

Easily, this universal property can be expressed as being a terminal object in a suitable category.

Claim: Product objects exist in the category of modules.

Proof: Let $(M_i)_{i\in I}$ be any family of modules, let $M = \prod_{i\in I} M_i$ (the direct product) and let $\pi_j = (m_i)_{i\in I} \mapsto m_j$ ("jth projection"). π_j is easily R-linear. Given $\pi'_j : M' \to M_j$, if a suitable $\alpha : M' \to M$ exists, then $\pi_j(\alpha(b)) = \pi'_j(b)$ for all $b \in M'$. The only possible choice is $\alpha(b) = (\pi'_i(b))_{i\in I}$, and easily such an α is a homomorphism. □

A **coproduct** of $(c_i)_{i\in I}$ is an object e together with an I-indexed family $(\rho_i : c_i \to e)_{i\in I}$ of morphisms such that for all objects e' and all families of morphisms $(\rho'_i : c_i \to e')_{i\in I}$, there is a unique $\alpha : e \to e'$ such that $\pi_i \circ \alpha = \pi'_i$, i.e. the following diagram commutes:



Theorem: Given $(M_i)_{i\in I}$, let $N = \bigoplus_{i\in I} M_i$ for each j and for each $b \in M_j$, $\rho_j(b)$ is a sequence which is 0 at $i \neq j$ and b at $i = j$. Then $(N, (\rho_j))$ is the coproduct of (M_i).

Proof: Next time. □