21-237: Math Studies Algebra I December 2, 2022

Lecture 35 : More on direct sums and cyclic modules

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1 Direct sums are coproducts

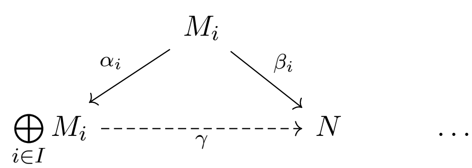
Recall: For R a ring, $(M_i)_{i\in I}$ a family of R-modules, the **direct sum** $\bigoplus_{i\in I} M_i$ was the set of families $(m_i \in M_i)_{i\in I}$ which are 0 almost everywhere.

For each $j \in I$, we can define $\alpha_j : M_j \to \bigoplus_{i\in I} M_i$ via $\alpha_j(b)_j = b$ and $\alpha_j(b)_i = 0$ for $i \neq j$.

Note: $\bigoplus_{i\in I} M_i$ is spanned by $\bigcup_{i\in I} \alpha_i[M_i]$.

Theorem: $\bigoplus_{i\in I} M_i$ equipped with (α_j) forms a coproduct in the category of R-modules.

Proof: We must show that given $(\beta_i : M_i \to N)_{i\in I}$ for some R-module N, there is a unique $\gamma : \bigoplus_{i\in I} M_i \to N$ such that for all $i \in I$, $\beta_i = \gamma \circ \alpha_i$, i.e. the following diagram commutes:



If γ exists, $\gamma((m_i)_{i\in I}) = \gamma(\sum_i \alpha_i(m_i)) = \sum_i \gamma(\alpha_i(m_i)) = \sum_i \beta_i(m_i)$, where we use our convention of "infinite sums" of sequences with finite supports and recalling the linearity of γ.

To finish, verify that such a γ is indeed R-linear, easy using the linearity of β, and that $\gamma \circ \alpha_i = \beta_i$, which is very easy from its definition. $\square$

2 Internal direct sums

Let N be an R-module. Let $M, M' \leq N$ such that

(1) $N = M + M' = \{m + m' : m \in M, m' \in M'\}$, which is also the least submodule containing $M \cup M'$.

(2) $0 = M \cap M'$

We say that N is the **internal direct sum** of M and M'.

Theorem: For every $n \in N$, there exists unique $m \in M$, $m' \in M$ such that $n = m + m'$.

Proof: Let $n \in N$. $N = M + M'$, so there exist $m \in M, m' \in M'$ such that $n = m + m'$.

For uniqueness, suppose $n = m_1 + m_1' = m_2 + m_2'$ for some $m_1, m_2 \in M, m_1', m_2' \in M'$. Then, $m_1 - m_2 = m_2' - m_1' \in M \cap M' = 0$, so $m_1 = m_2$ and $m_1' = m_2'$. $\square$

Corollary: $N \simeq M \oplus M'$. We can tediously verify that the bijection between N and ordered pairs from $M \oplus M'$ from above is an isomorphism.

More generally: If N is an R-module, $M_1, \ldots, M_k$ are submodules and every element of N is of the form $m_1 + m_2 + \ldots m_k$, $m_i \in M_i$ for unique $m_1, \ldots, m_k$, then we say M is the **internal direct sum** of $M_1, \ldots, M_k$. Equivalently, the following two conditions hold:

(a) $N = M_1 + \ldots + M_k$

(b) For all $(m_1, \ldots, m_k)$, $m_i \in M_i$, if $\sum_{i=1}^k m_i = 0$, then $m_i = 0$ for all i.

3 Cyclic modules

Let R be a ring, M an R-module. We say M is **cyclic** if there is $m \in M$ such that $M = Rm$.

Let's examine the properties of cyclic modules. Suppose that $M = Rm$. Let $\phi : R \to M$ via $\phi : r \mapsto rm$. ϕ is a linear map of R-modules, when R is viewed as an R-module. ϕ is surjective because $M = Rm$.

Let $I = \ker(\phi)$. Then, ϕ induces an isomorphism from R/I (considered as an R-module) to M.

$I = \{r \in R : \forall m \in M, rm = 0\}$ is called the **annihilator** of M, or $\mathrm{Ann}_R(M)$.

Conversely, if I is an ideal of R, then viewing R/I as an R-module, $R/I = R(1 + I)$, so R/I is cyclic.