21-237: Math Studies Algebra I December 5, 2022

Lecture 36 : Noetherian quotient properties

Lecturer: James Cummings

Scribe: Rajeev Godse

1 Polynomials and freely generated rings

Let R, S be rings, with $R \leq S$. Given $a_1, \ldots, a_n \in S$, consider the least subring of S containing $R \cup \{a_1, \ldots, a_n\}$. Written $R[a_1, \ldots, a_n]$, it's easy to see that it's equal to $\{f(a_1, \ldots, a_n) : f \in R[x_1, \ldots, x_n]\}$. It's also easy to see that the "evaluation map" $\phi : f \mapsto f(a_1, \ldots, a_n)$ is a HM from $R[x_1, \ldots, x_n]$ to S with $\operatorname{im}(\phi) = R[a_1, \ldots, a_n]$. By the first isomorphism theorem, $R[x_1, \ldots, x_n] \simeq R[x_1, \ldots, x_n]/\ker(\phi)$.

Intution: $R[x_1, \ldots, x_n]$ is the "most general" ring of the form $R[a_1, \ldots, a_n]$.

2 The N-word

Recall: An R-module M is Noetherian $\iff$ all submodules of M are finitely generated $\iff$ all increasing chains of submodules are eventually constant.

R is a Noetherian ring $\iff$ R is a Noetherian R-module $\iff$ all ideals are finitely generated $\iff$ all increasing chains of ideals are eventually constant.

R is Noetherian $\implies$ $R[x]$ is Noetherian, and by an easy inductive argument $R[x_1, \ldots, x_n]$ is Noetherian (as $R[x_1, \ldots, x_{n-1}][x_n] \simeq R[x_1, \ldots, x_n]$).

Fact: If R is Noetherian and I is an ideal of R, then R/I is Noetherian.

Proof: The ideals of R/I correspond to the ideals of R containing I, so an ideal of R/I must be of the form J/I. Then, say finite S generates J, then $\phi_I[S]$ generates J/I. $\square$

Corollary: If R is Noetherian, $R \leq S$, $a_1, \ldots, a_n \in S$, then $R[a_1, \ldots, a_n]$ is Noetherian.

Fact: Let R be any ring. Let N be an R-module, $M \leq N$. The following are equivalent:

(1) N is a Noetherian R-module.

(2) M and N/M are both Noetherian R-modules.

Proof: Suppose N is Noetherian. Any increasing chain of submodules of M is an increasing chain of submodules of N, hence it's eventually constant. Submodules of N/M are in correspondence with submodules of N containing M, so any increasing chain of submodules in N/M corresponds to an increasing chain of submodules of N containing M, which stabilizes, so its quotient does as well. Thus, $M, N/M$ are Noetherian.

Conversely, assume that $M, N/M$ are Noetherian. Let $N_0 \leq N_1 \leq N_2 \leq \ldots$ be submodules of N.

$(N_i \cap M)_{i \in \mathbb{N}}$ is an increasing chain of submodules of M. As M is Noetherian, it is eventually constant.

$\phi_M[N_i] = (N_i + M)/M$, and $(\phi_M[N_i])_{i \in \mathbb{N}}$ is an increasing chain of submodules of N/M. As N/M is Noetherian, it is eventually constant.

So there is $i \in \mathbb{N}$ such that for $j \geq i$, $N_j \cap M = N_i \cap M$ and $\phi_M[N_i] = \phi_M[N_j]$.

Claim: $N_i = N_j$ for $j \geq i$.

Proof: Since $N_i \subseteq N_j$, it suffices to show $N_j \subseteq N_i$.

Let $n \in N_j$. $\phi_M(n) \in \phi_M[N_j] = \phi_M[N_i]$. So there is $\overline{n} \in N_i$ such that $n + M = \overline{n} + M$, i.e. $\overline{n} - n \in M$. As $N_i \subseteq N_j$, $n, \overline{n} \in N_j$, so $\overline{n} - n \in M \cap N_j = M \cap N_i \subseteq N_i$. So $n = \overline{n} - (\overline{n} - n) \in N_i$. $\square$

So indeed, N is Noetherian under this hypothesis. $\square$

Corollary 1: If M, N both Noetherian R-modules, then $M \oplus N$ is Noetherian.

Proof: $M \oplus 0 \simeq M$, $\frac{M \oplus N}{M \oplus 0} \simeq N$. $\square$

Corollary 1.5: If $M_1, \ldots, M_t$ are Noetherian, $\bigoplus_{j=1}^{t} M_j$ Noetherian. *Proof*: Induction, Corollary 1. $\square$

Corollary 2: If R is a Noetherian ring, then the R-module R^n is a Noetherian R-module.

Proof: It's the direct sum of n copies of R. Apply Corollary 1.5.

Corollary 3: If R is a Noetherian ring and M is a finitely generated R-module, then M is a Noetherian R-module.

Proof: Let M be spanned by $m_1, \ldots, m_n$, let $\phi : R^n \to M$, $\phi(r_1, \ldots r_n) = \sum_{i=1}^{n} r_i m_i$, ϕ is surjective and R-linear.

Apply the first IM theorem to see that $M \simeq R^n / \ker(\phi)$. R^n is Noetherian from corollary 2, and then apply the theorem. $\square$