21-237: Math Studies Algebra I December 7, 2022

Lecture 37 : Adjunction

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1 Categorical constructions

Let $\mathcal{C}$ be a category. $\mathcal{C}^{op}$ is "$\mathcal{C}$ with the arrows reversed."

For categories $\mathcal{C}, \mathcal{D}$, define $\mathcal{C} \times \mathcal{D}$, the **product** of $\mathcal{C}$ and $\mathcal{D}$ with objects (c, d) for c an object of $\mathcal{C}$ and d an object of $\mathcal{D}$ and morphisms $(f, g) : (c_1, d_1) \to (c_2, d_2)$ for arrows $f : c_1 \to c_2$ in $\mathcal{C}$, $g : d_1 \to d_2$ in $\mathcal{D}$.

For categories $\mathcal{C}, \mathcal{D}$, define $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, the **functor category** of $\mathcal{C}, \mathcal{D}$ with objects functors $F : \mathcal{C} \to \mathcal{D}$ and arrows $\eta : F \to G$ natural transformations from F to G, i.e. a family $(\eta_c : Fc \to Gc)_{c \in \mathrm{obj}(\mathcal{C})}$ such that for all arrows $\alpha : c \to c'$ in $\mathcal{C}$, $G\alpha \circ \eta_c = \eta_{c'} \circ F\alpha$, i.e. the following diagram commutes:

$$\begin{CD} Fc @>{F\alpha}>> Fc' \\ @V{\eta_c}VV @VV{\eta_{c'}}V \\ Gc @>>{G\alpha}> Gc' \end{CD}$$

Given a category $\mathcal{C}$, objects c, c' of $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(c, c')$ is the collection of arrows from c to c' in $\mathcal{C}$.

A natural question to ask is how we might compare $\mathrm{Hom}_{\mathcal{C}}(c, c')$ and $\mathrm{Hom}_{\mathcal{C}}(d, d')$.

It's category theory! Let's use arrows.

For some $\alpha : d \to c$, $\beta : c' \to d'$, we can map $\gamma : c \to c'$ to $\beta\gamma\alpha : d \to d'$.

If $\mathcal{C}$ is locally small, we can construct the **hom functor**, $\mathrm{Hom}_{\mathcal{C}}(-, -) : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$.

For object (c, c'), $\mathrm{Hom}(c, c')$ is the set we defined earlier.

For morphism $(\alpha, \beta) : (c, c') \to (d, d')$, $\mathrm{Hom}(\alpha, \beta)(\gamma) = \beta\gamma\alpha$, typechecks as above.

Also note that in any category $\mathcal{C}$, an arrow $\alpha : c \to c'$ is an **isomorphism** if there is an arrow $\beta : c' \to c$ such that $\alpha\beta = \mathrm{id}_{c'}$ and $\beta\alpha = \mathrm{id}_c$. If β exists, it's unique, and we write $\beta = \alpha^{-1}$.

2 Adjunctions

Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be functors. Consider $\mathrm{Hom}_{\mathcal{D}}(F-, -)$ as a functor with domain $\mathcal{C}^{op} \times \mathcal{D}$ and $\mathrm{Hom}_{\mathcal{C}}(-, G-)$ as a functor with domain $\mathcal{C}^{op} \times \mathcal{D}$.

An **adjunction** between F and G is a natural isomorphism between $\mathrm{Hom}_{\mathcal{C}}(-, G-)$ and $\mathrm{Hom}_{\mathcal{D}}(F-, -)$.