21-237: Math Studies Algebra I December 9, 2022

Lecture 38 : Adjoints preserve limits

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1 Adjunction in action

Let Fr : **Set** $\to$ **Grp** be the *free group functor*, i.e. for set X, $\mathrm{Fr}(X)$ is the free group on X, and for function $f : X \to Y$, $\mathrm{Fr}(f)$ is unique HM ϕ such that $\mathrm{Fr}(X) \to \mathrm{Fr}(Y)$ such that $\phi(x') = f(x)'$ for $x' \in X$.

Let Un : **Grp** $\to$ **Set** be the *forgetful functor*, i.e. $\mathrm{Un}(G, \cdot) = G$ and $\mathrm{Un}\phi = \phi$.

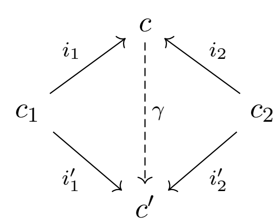
Notice: For all sets X and groups G, there is a natural bijection between $\mathrm{Hom}_{\mathbf{Grp}}(\mathrm{Fr}(X), G) = \{\phi : \mathrm{Fr}(X) \to G \mid \phi \text{ HM}\}$ and $\mathrm{Hom}_{\mathbf{Set}}(X, \mathrm{Un}(G)) = \{f : X \to G \mid f \text{ function}\}$.

So Fr and Un are adjoint.

2 Right (left) adjoints preserve (co)limits

Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$, $F \dashv G$ as witnessed by $\nu_{(c,d)} : \mathrm{Hom}_{\mathcal{D}}(Fc, d) \leftrightarrow \mathrm{Hom}_{\mathcal{C}}(c, Gd)$.

Let c_1, c_2 be objects of c which have a coproduct c with injection maps $i_1 : c_1 \to c, i_2 : c_2 \to c$ such that for all c', $i'_1 : c_1 \to c'$, $i'_2 : c_2 \to i'_2$, there is unique $\gamma : c \to c'$ such that $\gamma \circ i_1 = i'_1$ and $\gamma \circ i_2 = i'_2$, i.e. the following diagram commutes:



Theorem: F preserves coproducts. That is, the coproduct of Fc_1 and Fc_2 is given by Fc with injections Fi_1, Fi_2. Note that this holds more generally for all colimits.

Proof: Take all the data in a coproduct diagram of Fc in $\mathcal{D}$. Applying the natural isomorphism, we get a coproduct diagram in $\mathcal{C}$. As c is a coproduct in $\mathcal{C}$, there is a unique $\gamma : c \to Gd$ such that $\gamma \circ i_1 = Gi'_1$ and $\gamma \circ i_2 = Gi'_2$.

Then, applying the natural isomorphism back to the world of $\mathcal{D}$, $\delta = F\gamma$ is the unique HM such that $\delta \circ Fi_1 = i'_1$ and $\delta \circ Fi_2 = i'_2$. $\square$