21-237: Math Studies Algebra I September 9, 2022

Lecture 5 : Homomorphisms and structure-respecting subgroups

Lecturer: James Cummings

Scribe (in absentia): Rajeev Godse

1 Homomorphisms and their accessories

Recall: $\phi : G_1 \to G_2$ is a **homomorphism** if $\phi(gg') = \phi(g)\phi(g')$ for all $g, g' \in G_1$.

Easy facts: $\phi(1) = 1$, $\phi(g)^{-1} = \phi(g^{-1})$

For HM ϕ, we define the **kernel** $\ker(\phi) = \{g \in G_1 : \phi(g) = 1\}$.

We further define the **image** $\text{im}(\phi) = \{g' \in g_2 : \exists g \in G_1.\, \phi(g) = g^{-1}\}$.

Easy facts: $\ker(\phi) \leq G_1$, $\text{im}(\phi) \leq G_2$.

Recall: An **automorphism** is an isomorphism from G to G, i.e. a permutation of G that is an HM. These are the *symmetries* of the group: permutations that respect the group structure.

Let G be a group, let $g, h \in G$. The **conjugate** of h by g, denoted h^g, is ghg^{-1}.

Theorem: If we define $\alpha_g : G \to G$ by $\alpha_g(h) = h^g$, $\alpha_g \in \text{Aut}(G)$.

Proof: The two-sided inverse of α_g is given by $\alpha_{g^{-1}}$: $(\alpha_{g^{-1}} \circ \alpha_g)(h) = g^{-1}ghg^{-1}g = h$, and $(\alpha_g \circ \alpha_{g^{-1}})(h) = gg^{-1}hgg^{-1} = h$ for all $h \in G$, so α_g is a permutation.

For $h_1, h_2 \in G$, $\alpha(h_1)\alpha(h_2) = gh_1g^{-1}gh_2g^{-1} = gh_1h_2g^{-1} = \alpha_g(h_1h_2)$.

Theorem: If we define $\alpha : G \to \text{Aut}(G)$ via $\alpha(g) = \alpha_g$, then α is an HM from G to $\text{Aut}(G)$.

Proof: $\alpha_{g_1g_2}(h) = g_1g_2hg_2^{-1}g_1^{-1} = \alpha_{g_2}(\alpha_{g_1}(h))$, so by extensionality, $\alpha_{g_1g_2} = \alpha_{g_1} \circ \alpha_{g_2}$.

Fact: A group G is abelian iff $h^g = h$ for all $g, h \in G$.

Notation: For $f : X \to Y$, $B \subseteq X$, we say $f[B] = \{y \in Y : \exists b \in B.\, f(b) = y\}$.

Easy fact: For a group G, $\alpha \in \text{Aut}(G)$, and $H \leq G$, $\alpha[H] \leq G$.

2 Special subgroups

Let G be a group, $H \leq G$. Then,

1. H is a **characteristic subgroup** of G, denoted H char G, iff $\alpha[H] = H$ for all $\alpha \in \text{Aut}(G)$
2. H is a **normal subgroup** of G, denoted $H \lhd G$, iff $H^g = \{h^g : h \in H\} = H$ for all $g \in G$.

Theorem: Let G be a group, $N \leq G$. The following are equivalent:

1. $N \lhd G$, i.e. $N^g = N$ for all $g \in G$
2. $gN = Ng$ for all $g \in G$
3. $N^g \leq N$ for all $g \in G$

Proof: (1) $\iff$ (2) is boring (left as an exercise), and (1) $\implies$ (3) is obvious. For (3) $\implies$ (1), see that $N^{g^{-1}} \leq N \iff N \leq N^g$.

Theorem: For HM $\phi : G_1 \to G_2$, $\ker(\phi) \lhd G_1$.

Proof: Let $g \in G_1$, $h \in \ker(\phi)$. $\phi(h^g) = \phi(ghg^{-1}) = \phi(g)\phi(h)\phi(g)^{-1} = \phi(g)1\phi(g)^{-1} = phi(g)\phi(g)^{-1} = 1$, so $h^g \in \ker(\phi)$. Then, $\ker(\phi)^g \leq \ker(\phi)$, so (3) is met.