21-237: Math Studies Algebra I September 12, 2022

Lecture 6 : Quotient groups

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1 Normal subgroups

Recall: For $N \leq G$, N is **normal** in G ($N \triangleleft G$) if any of the following hold:

1. $N^g = N$ for all $g \in G$
2. $gN = Ng$ for all $g \in G$
3. $N^g \leq N$ for all $g \in G$

2 Quotient groups

The homomorphism associated with the quotient group is often more important than the quotient group itself. As category theorists say, it's the arrows that matter.

Let G be a group, let $N \triangleleft G$. We define the **quotient group** G/N, along with **quotient homomorphism** $\phi_N : G \to G/N$.

The underlying set of G/N is the set of cosets of N on G (since $N \triangleleft G$, $gN = Ng$, i.e. the left and right cosets coincide).

We define the binary operation $(g_1N)(g_2N) = (g_1g_2)N$.

We verify this gives a well-defined binary operation:

Note: For any $H \leq G$, $aH = bH \iff a = bh$ for some $h \in H \iff ah = b$ for some $h \in H \iff b^{-1}a \in H \iff a^{-1}b \in H$.

Let $g_1N = g_1'N$, $g_2N = g_2'N$. From the note, $g_1^{-1}g_1' \in N$, $g_2^{-1}g_2' \in N$.

$$\begin{aligned} g_2^{-1}g_1^{-1}g_1'g_2' &= g_2^{-1}g_1^{-1}g_1'g_2g_2^{-1}g_2' \\ &= (g_1^{-1}g_1')^{g_2^{-1}}(g_2^{-1}g_2') \in H \qquad \text{(Closure under product, conjugation)} \end{aligned}$$

By the note, $g_1g_2H = g_1'g_2'H$. So the binary operation we defined is a well-defined function.

Claim 1: the quotient group is a group

Proof: $((g_1N)(g_2N))(g_3N) = ((g_1g_2)N)(g_3N) = ((g_1g_2)g_3)N = (g_1(g_2g_3))N = (g_1N)((g_2g_3)N) = (g_1N)((g_2N)(g_3N))$, so the operation is associative.

$(1N)(gN) = gN = (gN)(1N)$, so $1N = N$ is the identity.

$(g^{-1}N)(gN) = (gg^{-1})N = N$, so $(gN)^{-1}$ is given by $g^{-1}N$.

Claim 2: ϕ_N, defined in the obvious way, is an HM

Proof: If we define $\phi_N = g \mapsto gN$, ϕ_N is an HM from G to G/N, from the equational reasoning above.

Claim 3: $N = \ker(\phi_N)$

Proof: $1_{G/N} = N$, so $\ker(\phi_N) = \{g : gN = N\} = N$ by extreme obviousness (N is a subgroup).

Upshot: $|G/N|$ = number of cosets of N in $G = [G : N]$. If G is finite, this is equal to $\frac{|G|}{|N|}$.

3 First Isomorphism Theorem

Theorem: Let $\phi : G_1 \to G_2$ be a HM. Let $N = \ker(\phi)$ (so $N \lhd G_1$). Then there is an isomorphism ψ from G_1/N to $\phi[G_1]$ given by $\psi = gN \mapsto \phi(g)$.

Key calculation: Let $g, g' \in G_1$. $\phi(g) = \phi(g') \iff \phi(g^{-1}g') = 1 \iff g^{-1}g' \in N \iff gN = g'N$.

Useful related fact: A HM ϕ is injective iff $\ker(\phi) = 1$.