21-237: Math Studies Algebra I September 14, 2022

Lecture 7 : Group actions

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1 First Isomorphism Theorem

Let $N \triangleleft G$. G/N is a **quotient group** on cosets of N in G with group operation $(g_1N)(g_2N) = g_1g_2N$.

This yields a natural HM $\phi_N : G \to G/N$, the **quotient homomorphism** via $\phi_N(g) = gN$. See that $\ker(\phi_N) = N$.

First Isomorphism Theorem: For HM $\phi : G_1 \to G_2$, let $N = \ker(\phi)$. Then, $\text{im}(\phi) \simeq G/N$ via $\psi(gN) = \phi(g)$.

Proof: For $g, g' \in G_1$, $\phi(g) = \phi(g') \iff \phi(g^{-1}g') = 1 \iff g^{-1}g' \in \ker(\phi) = N \iff gN = g'N$. Therefore, ψ is a bijection.

$\psi((gN)(g'N)) = \psi((gg')N) = \phi(gg') = \phi(g)\phi(g') = \psi(gN)\psi(g'N)$, so ψ is HM. So ψ is IM.

1.1 Related facts

Fact: For a group G, $K_1, K_2 \leq G$, $K_1 \subseteq K_2 \iff K_1 \leq K_2$.

Fact: For a group G, $N \triangleleft G$, the subgroups of G/N are in bijection with $\{H : N \leq H \leq G\}$. In this bijection, H/N corresponds to H (makes sense because if $N \triangleleft G$, $N \leq H \leq G$, then $N \triangleleft H$).

2 Group actions

2.1 Definition

Let G be a group and let X be a set. An **action** of G on X is a function[1] from $G \times X = \{(g, x) : g \in G, x \in X\}$ to X satisfying the following axioms:

1. $1 \cdot x = x$ for all $x \in X$.
2. $g_1 \cdot (g_2 \cdot x) = (g_1g_2) \cdot x$ for all $g_1, g_2 \in G$, $x \in X$.

Let G act on X. For $x \in X$, the **stabilizer** G_x is $\{g : G : g \cdot x = x\}$, and the **orbit** O_x is $\{g \cdot x : g \in G\}$.

2.2 Examples

1. Let G be any group, let $X = \{H : H \leq G\}$, and let $g \cdot H = H^g$.
2. Let $[n] = \{1, \ldots, n\}$, $[n]^k = \{A \subseteq [n] : |A| = k\}$. Recall that S_n is the group of permutations on $[n]$. Let $\sigma \cdot A = \{\sigma(j) : j \in A\}$.

2.3 Equivalence relation

Say $x \sim y \iff \exists g \in G.\, g \cdot x = y$. This is an ER on x:

1. Reflexivity: let $g = 1$ and apply axiom 1.

[1] $g \cdot x$ is the value of the function on (g, x)

2. Symmetry: If $g \cdot x = y$, then $g^{-1} \cdot y = g^{-1} \cdot (g \cdot x) = (g^{-1}g) \cdot x = 1 \cdot x = x$.

3. Transitivity: follows from axiom 2.

The equivalence class of x with respect to $\sim$ is O_x.

Note: $G_x \leq G$. The proof follows the structure of the ER proof above.

2.4 Orbit Stabilizer Theorem

Let $g_1, g_2 \in G$ and $x \in X$. Then, $g_1 \cdot x = g_2 \cdot x \iff (g_2^{-1} g_1) \cdot x = x \iff g_2^{-1} g_1 \in G_x \iff g_1 G_x = g_2 G_x$.

Theorem: There is a bijection between the left cosets of G_x and points in the orbit of x, in which $gG_x \leftrightarrow g \cdot x$. Thus, if G is finite, $|O_x| = [G : G_x] = \frac{|G|}{|G_x|}$.