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Lecture 8 : Conjugation action

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1 Group actions as HMs

Notice that given a HM $\phi : G \to \Sigma_X$, $g \cdot x = \phi(g)(x)$ is easily a group action.

Conversely, any action arises in this way: define $\phi(g) : X \to X$ via $\phi(g)(x) = g \cdot x$. $\phi(g)$ and $\phi(g^{-1})$ are mutually inverse, so $\phi(g) \in \Sigma_X$. Some equational reasoning shows that $\phi(g_1 g_2) = \phi(g_1) \circ \phi(g_2)$.

2 Conjugation

For a group G, we define the **conjugation action** of G on G by $g \cdot h = h^g = ghg^{-1}$: for $h \in H$, $1 \cdot h = 1h1 = h$, and for $g, g' \in G$, $(h^{g'})^g = gg'hg'^{-1}g^{-1} = gg'h(gg')^{-1} = h^{gg'}$.

Recall: In general, $O_x = \{g \cdot x : g \in G\}$, which is the equivalence class of x under the ER where $x \sim y \iff \exists g.\, g \cdot x = y$.

In the case of the conjugation action, for $h \in G$, $O_h = \{h^g : g \in G\}$, which is the equivalence class of h under the ER of conjugacy (h_1 conjugate to $h_2 \iff \exists g.\, h_1^g = h_2$).

Again recall: In general, $G_x = \{g \in G : g \cdot x = x\}$, and $G_x \leq G$.

In the case of the conjugation action, stabilizer of h is $\{g : h^g = h\} = \{g : gh = hg\}$, also known as the centralizer of h, denoted $C_G(h)$.

Continue recalling: In general, left cosets of G_x correspond bijectively to points in the orbit of x, where $gG_x \leftrightarrow g \cdot x$. This works since $g \cdot x = h \cdot x \iff gG_x = hG_x$.

In the case of the conjugation action, group elements in the conjugacy class of h correspond bijectively with left cosets of $C_G(h)$, where $gC_G(h) \leftrightarrow ghg^{-1}$.

Now, assume G is finite. In general, $|O_x| = [G : G_x] = \frac{|G|}{|G_x|}$.

For conjugacy classes, $|\text{conjugacy class of } h| = \frac{|G|}{|C_G(h)|}$.

Note the class equation, $|G| =$ sum of the sizes of conjugacy classes of G, which follows easily from the fact that conjugacy is an equivalence relation.

Let G be a group. The **center** of G is the subgroup given by $Z(G) = \{h : \forall g \in G.\, gh = hg\}$.

Useful facts

1. $Z(G)$ is a characteristic subgroup of G, i.e. $\alpha[Z(G)] = Z(G)$ for all $\alpha \in \mathrm{Aut}(G)$.
2. If $N \leq Z(G)$, then $N \lhd G$. *Proof*: $N^g = \{n^g : n \in N\} = \{n : n \in N\} = N$.

Given an action of G on X, we say $x \in X$ is a **fixed point** if $g \cdot x = x$ for all $g \in G$, or equivalently, $O_x = \{x\}$.

In the case of the conjugation action, $h \in Z(G) \iff h^g = h$ for all $g \in G \iff h$ is a fixed point of the conjugation action $\iff$ the conjugacy class of h is $\{h\}$.

Theorem: Let p be prime, let G be a group, where $|G| = p^n$ for $n > 0$. Then, $Z(G) \neq 1$.

Proof: If C is any class and $h \in C$, $|C| = \frac{|S|}{|C_G(h)|} = p^k$ for some k. By the class equation, $|G|$ is a sum of powers of p. And the conjugacy class of 1 has size 1, so there must be some other conjugacy class of size 1.