

# A Logical Framework with Infinitary Terms

ZHIBO CHEN, Carnegie Mellon University, USA

Logical frameworks are successful in modeling proof systems. Recently, CoLF extended the logical framework LF to support higher-order rational terms that enable adequate encoding of circular objects and derivations. In this paper, we propose  $\text{CoLF}^\omega$  as an alternative interpretation of CoLF-style signatures where terms are taken to be all possibly infinitary terms that are consistent with a given signature. In particular, we propose the notion of productive Böhm trees, a particular kind of typed  $\perp$ -free Böhm trees that are closed under hereditary substitution. We show that the productive Böhm trees are capable of meta-encoding their own structure. Overall, we hope to establish  $\text{CoLF}^\omega$  as a new formal framework for the encoding of infinitary regular and non-regular structures.

## 1 INTRODUCTION

Infinite objects are representable in the logical framework LF by indexing a type family with a natural number as its observation depth. For example, in the following signature, the stream of natural numbers whose first  $k$  elements can be observed is in compositional bijection with the canonical terms of the type family  $\text{stream}(\text{succ}^k \text{ zero})$ .

```

nat : type.
zero : nat.
succ : nat -> nat.
stream : nat -> type.
unobservable : stream zero.
cocons : {k : nat} nat -> stream k -> stream (succ k).

```

The encoding is hard to work with, because the observation depth of the stream needs to be tracked everywhere a stream is used. CoLF [Chen and Pfenning 2023] is an extension of the logical framework LF that supports natural and adequate encodings of circular objects and circular derivations. To make type checking decidable, CoLF limits its term model to higher-order rational terms. This limitation has the shortcoming that objects without a regular structure cannot be adequately represented in CoLF. For example, the stream of natural numbers with repeating 1's and 2's,  $1, 2, 1, 2, \dots$ , can be encoded in CoLF because it has a regular structure, i.e. the stream can be given by the equation  $S = 1, 2, S$ . The stream of natural numbers counting up from 1,  $1, 2, 3, 4, \dots$ , cannot be encoded in CoLF because it does not have a regular structure, i.e. the stream cannot be given by a system of equations. In this paper, we develop a new type theory  $\text{CoLF}^\omega$ , which provides an alternative term model for CoLF-style signatures where terms are taken to be all possibly infinitary terms. Many more interesting infinitary objects can be encoded in  $\text{CoLF}^\omega$  than in CoLF.

The main contributions of this paper are:

- A formulation of infinitary syntax trees (Section 3).
- A definition of productive Böhm trees via the infinitary syntax trees (Section 4).
- The type theory of  $\text{CoLF}^\omega$ , whose terms are productive Böhm trees (Section 5).
- An interpretation of (adapted) finitary signatures of CoLF into  $\text{CoLF}^\omega$  (Section 6).
- A meta-encoding of the productive Böhm trees using  $\text{CoLF}^\omega$  signatures (Section 7).
- A case study on co-natural numbers and co-binary numbers using  $\text{CoLF}^\omega$  (Section 8).

## 2 EXAMPLES OF $\text{CoLF}^\omega$

We illustrate informally the infinitary term model of  $\text{CoLF}^\omega$ , and how it is different from the rational term model of CoLF.

## 2.1 Streams

Consider the following CoLF signature for defining streams of natural numbers.

```

nat : type.
zero : nat.
succ : nat -> nat.
stream : cotype.
cocons : nat -> stream -> stream.

```

In CoLF, the only terms are *rational terms*, i.e. terms having finitely many subterms up to renaming of free and bound variables. Canonical terms of type `stream` are rational. As a consequence, we can only represent rational streams (streams with finitely many distinct repeating patterns) in the framework. A stream that counts up from a certain natural number or a stream that enumerates all Fibonacci numbers is not a term of type `stream` in CoLF. However, in  $\text{CoLF}^\omega$ , all streams are infinitary terms consistent with the signature. That is, the canonical terms of type `stream` include all possible streams, and there are uncountably many of them.

There is a question of whether noncomputable streams are also represented in the canonical terms. For example, temperature readings from a measurement device can be a stream of natural numbers that is not computable. We choose to leave open the question of computability intentionally in the hope that the development of  $\text{CoLF}^\omega$  can be used to encode either computable or noncomputable objects, as long as the choice is made consistently.

While the canonical terms of type `stream` can be any stream, we may specify the streams we actually care about using predicates. For instance, we can specify a stream that counts up from the natural number  $n$  by saying that the stream  $S$  is a term of type `stream` such that  $\text{up } n \ S$  holds, where  $\text{up}$  is the predicate defined below.

```

up : nat -> stream -> cotype.
up/def : {N : nat} {S : stream} up (succ N) S
        -> up N (cocons N S).

```

A term of type  $\text{up } N \ S$  must be a term of the form  $\text{up/def } N \ (\text{cocons } N \ S') \ U$ , where  $S = \text{cocons } N \ S'$ , and  $U$  is a term of type  $\text{up } (\text{succ } N) \ S'$ . In fact, there is a unique inhabitant of type  $\text{up } N \ S$  for each  $N$ . We show an example of a term of type  $\text{up } \text{zero} \ S$  where  $S$  is required to be a stream that counts up from 0. To reduce visual clutter, we write  $N :: S$  for  $\text{cocons } N \ S$ , 0 for `zero`, 1 for `succ zero`, etc. We have

```

up/def 0 S : up 1 S -> up 0 (0 :: S).
up/def 1 S : up 2 S -> up 1 (1 :: S).
up/def 2 S : up 3 S -> up 2 (2 :: S).
...

```

Then,

```

m : up 0 (0 :: 1 :: 2 :: 3 :: ... ) =
  up/def 0 (1 :: 2 :: 3 :: ...)
    (up/def 1 (2 :: 3 :: ...)
      (up/def 2 (3 :: ...)
        (up/def 3 ... ...
          )
        )
      )
    ).

```

The term  $m$  is not typeable in CoLF exactly because it is not a rational term: the set of its subterms contains  $\text{succ}^n \ \text{zero}$  for every natural number  $n$ .

## 2.2 Fibonacci Sequences

We could define complex infinitary streams where the later parts of the stream depend on earlier parts. Let  $\text{fib } n \ m \ S$  denote the stream of the Fibonacci sequence whose previous two numbers are  $n$  and  $m$ . The whole Fibonacci stream starting with 1 would be the stream  $S$  such that the type  $\text{fib zero (succ zero) } S$  is inhabited.

```
fib : nat -> nat -> stream -> cotype.
fib/def : add X Y Z
  -> fib Y Z S
  -> fib X Y (cocons Z S).
```

The  $\text{add } X \ Y \ Z$  predicate is defined inductively and is inhabited if  $X + Y = Z$ .

## 2.3 Real Numbers

A bit stream  $b_0, b_1, \dots$  can represent the binary expansion of a real number in  $[0, 1]$

$$b_0, b_1, \dots \rightsquigarrow \sum_{i=0}^{\infty} (b_i \cdot \frac{1}{2^{i+1}})$$

```
bitstream : cotype.
b0 : bitstream -> bitstream.
b1 : bitstream -> bitstream.
```

For example, the real number  $0.101010\dots$  (binary decimal expansion) can be represented by the bit stream  $1, 0, 1, 0, 1, 0, \dots$  whose encoding in  $\text{CoLF}^\omega$  is  $\text{b1 (b0 (b1 (b0 (b1 (b0 \dots))))))}$ . Note that this real number is actually a rational number and thus is representable in  $\text{CoLF}$  as the term

```
n : bitstream = b1 (b0 n).
```

An example of an irrational real number would be the number,  $0.1010010001\dots$ , and it can be represented in  $\text{CoLF}^\omega$  as the following infinitary term.

```
m : bitstream = b1 (b0 (b1 (b0 (b0 (b1 (b0 (b0 (b0 (b1 \dots))))))))).
```

This number is not representable in  $\text{CoLF}$  because the corresponding term is not a rational term. It is easy to see that rational numbers correspond to rational terms, and irrational numbers correspond to irrational terms. In summary,  $\text{CoLF}^\omega$  can represent all real numbers<sup>1</sup> whereas  $\text{CoLF}$  can only represent rational ones between 0 and 1.

## 3 INFINITARY SYNTAX TREES

We give an account for infinitary syntax trees, which serve as the technical device for defining productive Böhm trees.

The concept of observation is central to infinitary structures. A finitary structure can be observed in its totality with a single observation, whereas an infinitary structure cannot be observed in its totality with a single observation. The number of remaining steps that a term may be observed is called the *observation depth* and is written using a number subscript in parentheses. Syntax categories will always have an observation depth attached. For example, we write  $M_{(k)}$  for a term with observation depth  $k$ , and  $A_{(k)}$  for a type with observation depth  $k$ . When we write down the grammar for a possibly infinitary term, the *infinitary grammar* will specify the cases when the term undergoes a single step of observation. Concretely, we specify the grammar for observation depth  $k + 1$  in terms of the grammar for terms of observation depth  $k$ . That is, given a syntax

<sup>1</sup>As with streams, we leave the issue of whether only computable reals are represented or all reals are represented as a decision that the user of  $\text{CoLF}^\omega$  can make. The reader is referred to [Bauer \[2005\]](#) for a discussion of the computability of real numbers.

category  $M$ , the grammar for  $M_{(k+1)}$  is specified in terms of  $M_{(k)}$  for coinductive definitions, and is specified in terms of  $M_{(k+1)}$  for inductive definitions. The grammar may be mutually recursive, in that the grammar for  $M_{(k+1)}$  may be specified in terms of  $M'_{(k)}$  or  $M'_{(k+1)}$  where  $M'$  is another syntax category. We also assume a universal base case with observation depth 0 for all syntax categories. For example, we write  $M_{(0)}$  for an unobservable term and  $A_{(0)}$  for an unobservable type. The depth  $\omega$  is used for non-finite depth. For example, we write  $M_{(\omega)}$  for a term that can be observed indefinitely and write  $A_{(\omega)}$  for a type that can be observed indefinitely. We may sometimes omit the depth  $\omega$  annotation for a syntax category, e.g. we may just write  $M$  for  $M_{(\omega)}$ , and  $A$  for  $A_{(\omega)}$ .

We illustrate our use of the infinitary grammar through a series of examples. The reader should be reminded that we are making infinitary structures directly and formally precise.

(1) Natural Numbers

Inductive grammars are used to define finitary structures. The inductive nature is exemplified by the fact that the grammar for all structures is defined at the current observation depth.

We have the definition for natural numbers  $N$ , transcribed from the usual inductive definition.

$$N_{(k+1)} ::= 0 \mid S N_{(k+1)}$$

The grammar specifies that in a single observation, a natural number  $N$  is either 0, or the successor of another natural number  $N'$ , where  $N'$  must be observed in the same observation. Given the finite nature of the observation, a natural number is a series of  $S$ 's followed by 0.

(2) Conatural Numbers

A slightly modified grammar defines the conatural numbers  $C$ .

$$C_{(k+1)} ::= 0 \mid S C_{(k)}$$

The grammar specifies that in a single observation, a conatural number  $C$  is either 0 or the successor of another conatural number  $C'$  where  $C'$  must be observed later, because it has one less observation depth.

(3) Bit streams

A stream of bits  $BS$  may be defined by the following grammar:

$$BS_{(k+1)} = b0, BS_{(k)} \mid b1, BS_{(k)}$$

The grammar specifies that an observation of a bitstream is either the zero bit  $b0$ , or the one bit  $b1$ , followed by another bitstream that must be observed in subsequent steps.

(4) Binary number

A binary number  $BN$ , is a bit stream of finite length, where the least significant bit is listed first. A binary number can be represented using the following grammar:

$$BN_{(k+1)} ::= b0, BN_{(k+1)} \mid b1, BN_{(k+1)} \mid \epsilon$$

The grammar specifies that an observation of a binary number will reveal that either it is empty, or a bit ( $b0$  or  $b1$ ) followed by another binary number that must be observed in the same observation. Because an observation may only reveal a finite amount of information, a binary number cannot have an infinite number of bits before  $\epsilon$ .

(5) Finitely-padded streams

A finitely-padded stream [Chen 2021; Chen and Pfenning 2023] (a.k.a. left-fair streams [Basold 2018]) is a stream of natural numbers with a finite number of padding between

any two numbers. The grammar below specifies a coinductive finitely-padded stream  $PS$  and an inductive padding  $P$  defined recursively.

$$PS_{(k+1)} ::= N_{(k+1)}, P_{(k+1)}$$

$$P_{(k+1)} ::= P_{(k+1)} \mid PS_{(k)}$$

The grammar specifies that a single observation on a finitely-padded stream will reveal it is a natural number, followed by a padding, both of which must be observed in the same observation. An observation on padding will reveal that it is either another padding, in which case this other padding must be further observed, or a finitely-padded stream, in which case the stream must be observed in the next observation. Overall, an observation on a stream will reveal it is a natural number followed by a finite amount of padding, and then followed by a stream that must be observed in the next observation.

(6) Different kinds of infinite  $\lambda$ -terms

[Kennaway et al. \[1997\]](#), and [Barendregt and Klop \[2009\]](#) observed three formulations of infinite lambda trees that have wide applications. They are Böhm trees ( $BT$ ), Lévy-Longo trees ( $LLT$ ), and Berarducci trees ( $BeT$ ). The essential difference is that Böhm trees may not contain infinite chains of applications or abstractions, Lévy-Longo trees may contain infinite chains of abstractions but not infinite chains of applications, and Berarducci trees may contain both infinite chains of abstractions and applications. All trees may not contain  $\beta$ -redexes.

Perhaps the easiest among the three is the grammar for Berarducci trees ( $BeT$ ) as specified below. The grammar is broken into canonical terms  $BeT$  and neutral terms  $BeT_{APP}$ . An observation of a  $BeT$  tree will reveal that it is  $\perp$ , or an abstraction, whose subterm shall be observed in the next step, or a neutral term that must be observed in the same observation, while an observation of a neutral  $BeT_{APP}$  tree will reveal that it is either a head variable, or an application where each subterm must be observed in a subsequent observation.

$$BeT_{(k+1)} ::= \perp \mid \lambda x. BeT_{(k)} \mid (BeT_{APP})_{(k+1)}$$

$$BeT_{APP_{(k+1)}} ::= x \mid (BeT_{APP})_{(k)} (BeT_2)_{(k)}$$

The grammar of Lévy-Longo trees differs from Berarducci trees in that if the observation reveals an application, the observation must continue into the argument subterm, thereby disallowing infinite chains of applications.

$$LLT_{(k+1)} ::= \perp \mid \lambda x. LLT_{(k)} \mid (LLT_{APP})_{(k+1)}$$

$$LLT_{APP_{(k+1)}} ::= x \mid (LLT_{APP})_{(k+1)} (LLT_2)_{(k)}$$

The grammar of Böhm trees has a further restriction that if the observation reveals  $\lambda$ -abstraction, then the observation must continue into its body, thereby disallowing infinite chains of abstractions. A single observation of a Böhm tree will reveal all abstractions, all applications, and finally the head variable.

$$BT_{(k+1)} ::= \perp \mid \lambda x. BT_{(k+1)} \mid (BT_{APP})_{(k+1)}$$

$$BT_{APP_{(k+1)}} ::= x \mid (BT_{APP})_{(k+1)} (BT_2)_{(k)}$$

In summary, infinitary syntax trees provide a formal foundation for infinitary structures, by stratifying an infinitary term into distinct chunks of observations. The distinct chunks are delineated through the concept of an observation depth.

### 3.1 Equality

We say that two potentially infinite syntax trees of observation depth  $k$  are equal *up to depth  $k$* , (notation  $=_{(k)}$ ) iff the observation of two terms up to depth  $k$  does not reveal a difference between those two terms. That is,  $M_{(k)} =_{(k)} M'_{(k)}$  if the first  $k$  observations of  $M$  and  $M'$  do not reveal a difference between them.

We always have the trivial case that  $M_{(0)} =_{(0)} M'_{(0)}$ , that is, two terms are trivially equal because the first zero observations of the two terms will not reveal a difference between them. Given an infinitary grammar, the equality at depth  $k + 1$  can always be defined structurally. As an example, given the grammar for conat,

$$C_{(k+1)} ::= 0 \mid SC_{(k)}$$

we define the equality by the following rules:

- (1) (Trivially)  $C_{(0)} =_{(0)} C'_{(0)}$
- (2)  $0 =_{(k+1)} 0$
- (3)  $SC_{(k)} =_{(k+1)} SC'_{(k)}$  if  $C_{(k)} =_{(k)} C'_{(k)}$ .

Here, the first rule says that two unobservable terms are equal up to depth 0. The second rule says that if an observation (on terms with depth  $k + 1$ ) reveals that both terms are zero, then they are equal up to depth  $k + 1$ . The third rule says that if an observation on terms with depth  $k + 1$  reveals that the left-hand side is the successor followed by a term  $C_{(k)}$  of depth  $k$ , and the right-hand side is the successor followed by a term  $C'_{(k)}$  of depth  $k$ , then the terms of depth  $k + 1$  are equal up to depth  $k + 1$  if  $C_{(k)}$  and  $C'_{(k)}$  are equal up to depth  $k$ .

One may wonder if equality could be defined on terms with different observation depths, and we answer that because of the nature of observation, a term of any observation depth may be viewed as a term of a lesser observation depth by definition. That is, given a term  $M_{(k)}$ , we can construct a term  $M_{(j)}$  with  $j < k$  that mimics the behavior of  $M_{(k)}$  for the first  $j$  steps. Therefore, the definition of equality on heterogeneous depths is not necessary.

As with the convention that we write  $M_{(\omega)}$  or simply  $M$  for terms of infinitary observation depth, we write  $=_{(\omega)}$  or simply  $=$  for equality relation on those infinitary terms.

## 4 PRODUCTIVE BÖHM TREES

The logical framework methodology establishes a bijective correspondence between the structures that we would like to encode and the terms of the logical framework. In the case of LF logical framework, deductions are represented by dependently-typed  $\lambda$ -terms [Harper and Licata 2007; Harper et al. 1993]. The dependently-typed  $\lambda$ -terms are just simply-typed  $\lambda$ -terms when the type annotation for  $\lambda$ -abstractions are erased [Watkins et al. 2002]. The simply-typed  $\lambda$ -terms have two crucial properties that make it a suitable target for a logical framework. First, every term has a  $\beta$ -normal- $\eta$ -long form, which provides a basis for term equality modulo  $\beta\eta$ -conversion. Second, the normal forms of the terms are closed under hereditary substitution, thereby enabling the higher-order encoding strategies. When infinitary structures become the target of the encoding, infinitary  $\lambda$ -terms become the natural choice for the term model of the logical framework.

None of the typed versions of the three kinds of infinitary  $\lambda$ -terms have our desired properties. First, they all contain the unsolvable term  $\perp$ , which has no place in the encoding of infinitary structures. Even if the  $\perp$  was removed from their structure, the term structures are not closed under hereditary substitution, (i.e. substitutions followed by  $\beta\eta$ -normalization). To see this, consider the term  $F = \lambda x. x (x (x (...)))$ , and  $F$  could be assigned the simple type  $(* \rightarrow *) \rightarrow *$ , where  $*$  is a base type. Let the term  $I$  denote the identity function,  $I = \lambda z. z : * \rightarrow *$ , we see that the term  $FI$ ,

or the substitution  $[F/y](y I)$ , is  $I^\omega = I(I(I \dots))$ . This term does not normalize to a head normal form [Barendregt and Klop 2009].

We formulate the notion of typed productive Böhm trees as a subclass of Böhm trees, with constants and without  $\perp$ , that are closed under hereditary substitution. First, we add constants (or constructors) to Böhm trees by fixing an infinite set of variable names to serve as constant names. Those variables are subsequently called constants (syntax category  $c$ ). When constructing a  $\lambda$ -abstraction, the binder name will never be one of the constant names, and constants never vary under substitution. We also adopt the notion of head-spine form [Watkins et al. 2002] for iterative applications. For example,  $x M_1 M_2 M_3$  is written  $x \cdot (M_1; M_2; M_3)$ . The infinitary grammar for productive Böhm trees is given below.

Canonical terms	$M_{(k+1)}, N_{(k+1)}$	$::=$	$\lambda x. M_{(k+1)} \mid R_{(k+1)}$
Neutral terms	$R_{(k+1)}$	$::=$	$x \cdot T_{(k+1)} \mid c \cdot S_{(k+1)}$
Continuing Spines	$T_{(k+1)}$	$::=$	$() \mid M_{(k+1)}; T_{(k+1)}$
Suspended Spines	$S_{(k+1)}$	$::=$	$() \mid M_{(k)}; S_{(k+1)}$

The difference between productive ( $\perp$ -free) Böhm trees and non-productive  $\perp$ -free Böhm trees that in a single observation of productive Böhm trees, the arguments following a variable head must be observed in the same observation whereas in non-productive  $\perp$ -free Böhm trees, the arguments are always observed in a subsequent observation. In other words, in productive Böhm trees, only when we encounter constants do we halt the current observation and defer the arguments to the next observation. The presence of constants gives rise to the notion of productivity, defined by a condition on the infinite traces.

A trace is a possibly infinite list of head variables or constants where each element is the head of a direct child of the preceding element. Formally, the set of possibly infinite traces of  $M_{(\omega)}$ ,  $\text{traces}(M_{(\omega)})$  is defined to be the following, where  $h$  is either a variable or a constant.

$$\text{traces}(\lambda x_1. \dots \lambda x_l. h \cdot (M_1; \dots; M_n)) = \begin{cases} \{h\} & \text{if } n = 0 \\ \{h, T \mid T \in \cup_i \{\text{traces}(M_i)\}\} & \text{if } n > 0 \end{cases}$$

We say that a term is *productive* if there are infinitely many occurrences of constants along each infinite trace of the term. If we observe along any trace in a productive infinite term, we will always encounter a constructor in a finite number of steps. There can be only finitely many variables between any two constants along any infinite trace through a productive term  $M$ . For example, the term  $Z = \lambda x. x \cdot (\lambda y. y \cdot (\lambda z. z \cdot (\dots)))$  is not productive as its only infinite trace is  $x, y, z, \dots$  consisting of only variables. The term  $F = \lambda x. x \cdot (x \cdot (x \cdot (\dots)))$  is not productive because its only trace, which is infinite, is  $x, x, x, \dots$ , and this trace consists of only variables. The term  $I = \lambda x. x$  is trivially productive because there are no infinite traces. The term  $\lambda x. c \cdot (c \cdot (c \cdot (\dots)))$  is productive because the trace  $c, c, c, \dots$  contains infinitely many occurrences of the constructors  $c$ . A similar reasoning shows that the term  $\lambda x. c \cdot (x \cdot (c \cdot (x \cdot (c \cdot (\dots)))))$  and the term  $\lambda x. c \cdot (x \cdot (c \cdot (x \cdot (x \cdot (c \cdot (x \cdot (x \cdot (x \cdot (c \cdot (\dots))))))))$  are both productive.

We show that the grammar for productive Böhm trees directly corresponds to the notion of productivity.

**THEOREM 4.1 (PRODUCTIVITY).** *We have*

- (1) *Every canonical term is productive.*
- (2) *Every  $\perp$ -free productive Böhm tree is a canonical term.*

**PROOF.** (1) Given a canonical term  $M_{(\omega)}$ , we show that there can only be finitely many variables between two constants on an infinite trace of  $M$ . For any  $c \cdot S$  which is a subterm of  $M$ , an observation (which is always finitary) of  $M$  will either involve  $S$  or not. If it involves  $S$ , another

constant has been encountered on this trace. If it does not involve  $S$ , then there is no infinitary trace because observations are always finitary.

(2) Given a Böhm tree  $T$  that is productive, we show that  $T$  can be stratified into distinct chunks of observations, and thereby  $T$  is a canonical term. Starting with the root of  $T$ , the first chunk of observation will be along all the traces starting with the root of them and ending with a constant. The traces must be finitary because of the productivity condition. The stratification can be repeated for each child term in the spines of the constants.  $\square$

#### 4.1 Hereditary Substitution

The notion of hereditary substitution [Harper and Licata 2007; Watkins et al. 2002] is used to define substitution on canonical terms in a typing-agnostic way. The definition of hereditary substitution does not require the argument terms to be well-typed in a dependently typed setting. In this way, we break the circular dependency between typing and substitution in a dependent type theory. The substitution is well-defined as long as a correct simple type of the argument term is provided.

The simple types  $\tau$  are inductively defined by the following grammar.<sup>2</sup>

$$\tau ::= * \mid \tau_1 \rightarrow \tau_2$$

The hereditary substitution  $[N_{(k)}/x]^\tau M_{(k)}$  is defined as long as  $\Delta \vdash N_{(k)} : \tau$ , where  $\Delta = h_1 : \tau_1, \dots, h_n : \tau_n$  is a mapping from constants and variables to their simple types. Here, head  $h$  refers to either  $c$  or  $x$ . The judgment is defined by induction on  $k$  and the structure of  $N$ .

$$\boxed{\Delta \vdash N_{(k)} : \tau}$$

$$\frac{}{\Delta \vdash N_{(0)} : \tau} \qquad \frac{\Delta, x : \tau_1 \vdash N_{(k+1)} : \tau_2}{\Delta \vdash \lambda x. N_{(k+1)} : \tau_1 \rightarrow \tau_2}$$

$$\frac{x : \tau_1 \rightarrow \dots \rightarrow \tau_n \rightarrow \tau \in \Delta \quad \forall_{i, 1 \leq i \leq n}. \Delta \vdash (N_i)_{(k+1)} : \tau_i}{\Delta \vdash x \cdot ((N_1)_{(k+1)}; \dots; (N_n)_{(k+1)}) : \tau}$$

$$\frac{c : \tau_1 \rightarrow \dots \rightarrow \tau_n \rightarrow \tau \in \Delta \quad \forall_{i, 1 \leq i \leq n}. \Delta \vdash (N_i)_{(k)} : \tau_i}{\Delta \vdash c \cdot ((N_1)_{(k)}; \dots; (N_n)_{(k)}) : \tau}$$

We extend hereditary substitution to infinite terms in the sense that given two infinite terms in their canonical form (i.e.  $\beta$ -normal- $\eta$ -long form), there is a systematic procedure of generating an infinite term that is the result of substituting one term (for free variables) into another. In particular, if the two input terms to the hereditary substitution procedure are of an observation depth  $k$ , then the resulting term can be calculated up to the observation depth  $k$ . The following judgments define hereditary substitution on productive Böhm trees. The type  $\tau$  in the judgment provides typing information for the term being substituted by  $(N_{(k)})$  or  $T_{(k)}$ .

$$\begin{array}{ll} [N_{(k)}/x]^\tau M_{(k)} =_{(k)} M'_{(k)} & \text{Hereditary substitution in canonical terms} \\ [N_{(k)}/x]^\tau R_{(k)} =_{(k)} R'_{(k)} & \text{Hereditary substitution in neutral terms} \end{array}$$

<sup>2</sup>Using the syntax tree described in this paper, the grammar definition should be understood as  $\tau_{(k+1)} ::= * \mid (\tau_1)_{(k+1)} \rightarrow (\tau_2)_{(k+1)}$ . In subsequent discussions when we say  $M_{(k+1)}$  has type  $\tau_1 \rightarrow \tau_2$ , it should be understood formally as  $M_{(k+1)}$  has type  $(\tau_1)_{(\omega)} \rightarrow (\tau_2)_{(\omega)}$ . For purely inductive definitions, the entire structure of the term can be revealed in a single observation, and we choose to omit the depth annotations completely for purely inductive definitions to reduce the visual clutter.



$[N_{(k)}/x]^\tau T_{(k)} =_{(k)} T'_{(k)}$	Hereditary substitution in continuing spines
$T_{(k)} \triangleright^\tau N_{(k)} =_{(k)} R'_{(k)}$	Continuing spine applications
$[N_{(k)}/x]^\tau S_{(k+1)} =_{(k+1)} S'_{(k+1)}$	Hereditary substitution in suspended spines

One feature of hereditary substitution worth noting is that even if the type  $\tau$  does not type the term being substituted for, the procedure still terminates but produces an undefined value, because no clauses will apply in the definition. In other words, the procedure of substitution is robust with respect to typing of the input terms. The typing information is to ensure that the procedure of hereditary substitution is well-defined as an inductive definition.

The judgments for hereditary substitution are defined by lexicographic induction on  $\tau$ ,  $k$  and the structure of the term on the right-hand side of  $\tau$  ( $M_{(k)}$ ,  $R_{(k)}$ ,  $T_{(k)}$ ,  $N_{(k)}$ , and  $S_{(k+1)}$  respectively) as follows.

$$\boxed{[N_{(k)}/x]^\tau M_{(k)} =_{(k)} M'_{(k)}}$$

$$[N_{(0)}/x]^\tau M_{(0)} =_{(0)} M'_{(0)}$$

$$[N_{(k+1)}/x]^\tau R_{(k+1)} =_{(k+1)} [N_{(k+1)}/x]^\tau R_{(k+1)}$$

$$[N_{(k+1)}/x]^\tau \lambda y. M_{(k+1)} = \lambda y. M'_{(k+1)}$$

$$\text{if } [N_{(k+1)}/x]^\tau M_{(k+1)} =_{(k+1)} M'_{(k+1)}$$

$$\boxed{[N_{(k)}/x]^\tau R_{(k)} =_{(k)} R'_{(k)}}$$

$$[N_{(0)}/x]^\tau R_{(0)} =_{(0)} R'_{(0)}$$

$$[N_{(k+1)}/x]^\tau (x \cdot T_{(k+1)}) =_{(k+1)} N'_{(k+1)}$$

$$\text{if } [N_{(k+1)}/x]^\tau T_{(k+1)} =_{(k+1)} T'_{(k+1)}$$

$$\text{and } T'_{(k+1)} \triangleright^\tau N_{(k+1)} =_{(k+1)} N'_{(k+1)}$$

$$[N_{(k+1)}/x]^\tau (y \cdot T_{(k+1)}) =_{(k+1)} y \cdot T'_{(k+1)}$$

$$\text{if } [N_{(k+1)}/x]^\tau T_{(k+1)} =_{(k+1)} T'_{(k+1)}$$

$$[N_{(k+1)}/x]^\tau (c \cdot S_{(k+1)}) =_{(k+1)} c \cdot S'_{(k+1)}$$

$$\text{if } [N_{(k)}/x]^\tau S_{(k+1)} =_{(k+1)} S'_{(k+1)}$$

$$\boxed{[N_{(k)}/x]^\tau T_{(k)} =_{(k)} T'_{(k)}}$$

$$[N_{(0)}/x]^\tau T_{(0)} =_{(0)} T'_{(0)}$$

$$[N_{(k+1)}/x]^\tau () =_{(k+1)} ()$$

$$[N_{(k+1)}/x]^\tau (M_{(k+1)}; T_{(k+1)}) =_{(k+1)} M'_{(k+1)}; T'_{(k+1)}$$

$$\text{if } [N_{(k+1)}/x]^\tau M_{(k+1)} =_{(k+1)} M'_{(k+1)}$$

$$\text{and } [N_{(k+1)}/x]^\tau T_{(k+1)} =_{(k+1)} T'_{(k+1)}$$

$$\boxed{T_{(k)} \triangleright^\tau N_{(k)} =_{(k)} R'_{(k)}}$$

$$T_{(0)} \triangleright^\tau N_{(0)} =_{(0)} R'_{(0)}$$

$$() \triangleright^* R_{(k+1)} =_{(k+1)} R_{(k+1)}$$

$$(N_{(k+1)}; T_{(k+1)}) \triangleright^{\tau_2 \rightarrow \tau_1} \lambda x. M_{(k+1)} =_{(k+1)} M''_{(k+1)}$$

$$\text{if } [N_{(k+1)}/x]^{\tau_2} M_{(k+1)} =_{(k+1)} M'_{(k+1)}$$

$$\text{and } T_{(k+1)} \triangleright^{\tau_1} M'_{(k+1)} =_{(k+1)} M''_{(k+1)}$$

$$\boxed{[N_{(k)}/x]^\tau S_{(k+1)} =_{(k+1)} S'_{(k+1)}}$$

$$[N_{(k)}/x]^\tau () =_{(k+1)} ()$$

$$\begin{aligned}
[N_{(k)}/x]^\tau(M_{(k)}; S_{(k+1)}) &=_{(k+1)} M'_{(k)}; S'_{(k+1)} \\
&\text{if } [N_{(k)}/x]^\tau M_{(k)} =_{(k)} M'_{(k)} \\
&\text{and } [N_{(k)}/x]^\tau S_{(k+1)} =_{(k+1)} S'_{(k+1)}
\end{aligned}$$

Note that in the substitution clauses for  $[N_{(k+1)}/x](c \cdot S_{(k+1)})$ , the premise assumes that the term  $N_{(k+1)}$  is of depth  $k$ . This is justified because, as we mentioned previously in Section 3.1, any term may be viewed at a lesser observation depth.

**THEOREM 4.2 (HEREDITARY SUBSTITUTION RESPECTS OBSERVATION DEPTH).** *If both  $M_{(k)}$  and  $N_{(k)}$  are terms of observation depth  $k$ , then for all  $\tau$ ,  $[N_{(k)}/x]^\tau M_{(k)}$  is of productive depth  $k$  if defined.*

**PROOF.** Straightforward lexicographic induction on  $\tau$ ,  $k$ , and the structure of  $M_{(k)}$ .  $\square$

**COROLLARY 4.3.** *If  $M$  (i.e.,  $M_{(\omega)}$ ) and  $N$  (i.e.  $N_{(\omega)}$ ) are infinitary productive terms, then so is  $[N/x]^\tau M$  (i.e.  $[N_{(\omega)}/x]^\tau M_{(\omega)}$ ).*

**PROOF.** The result of hereditary substitution can have arbitrary productive depth by the previous proof.  $\square$

**THEOREM 4.4 (COMMUTATION OF HEREDITARY SUBSTITUTION).** *For all  $k$ , if  $[(N_1)_{(k)}/x]^{\tau_1} [(N_2)_{(k)}/z]^{\tau_2} M_{(k)} =_{(k)} M'_{(k)}$ , then  $[[N_1]_{(k)}/x]^{\tau_1} [N_2]_{(k)}/z]^{\tau_2} [[N_1]_{(k)}/x]^{\tau_1} M_{(k)} =_{(k)} M'_{(k)}$ .*

**PROOF.** By lexicographic induction on  $k$ ,  $\tau_2$ , and the structure of  $M$ .  $\square$

## 5 THE TYPE THEORY OF COLF $^\omega$

We present the type theory of a logical framework whose term model is typed productive Böhm trees.

### 5.1 Syntax

Besides the canonical terms, a logical framework also has the syntactic classes of canonical and atomic types, kinds, signatures, and contexts. We use “expressions” to refer to these entities (terms, types, kinds) in general. Expressions may contain potentially infinite terms with observation depths, and so they are also parametrized by observation depths. Although the terms may be infinitary, the structure of kinds and types are finitary. Thus, two expressions are equal up to depth  $k$  if they are structurally equal and the underlying terms are equal up to depth  $k$ . We write  $A_{(k)} \rightarrow B_{(k)}$  for  $\Pi x : A_{(k)}. B_{(k)}$  if  $x$  does not occur in  $B_{(k)}$ . Similarly, we may write  $A_{(k)} \rightarrow K_{(k)}$  for  $\Pi x : A_{(k)}. K_{(k)}$ . The syntax for signatures, contexts, kinds, and types are as follows. Notice that the depth remains  $k + 1$  on the right-hand side of all grammar rules for types and kinds, i.e. the type and kind structures are essentially inductive.

Signature	$\Sigma_{(k+1)}$	$::=$	$\cdot \mid \Sigma, a : K_{(k+1)} \mid \Sigma, c : A_{(k+1)}$
Context	$\Gamma_{(k+1)}$	$::=$	$\cdot \mid \Gamma_{(k+1)}, x : A_{(k+1)}$
Kind	$K_{(k+1)}$	$::=$	<b>type</b> $\mid$ <b>cotype</b> $\mid \Pi x : A_{(k+1)}. K_{(k+1)}$
Canonical types	$A_{(k+1)}, B_{(k+1)}$	$::=$	$P_{(k+1)} \mid \Pi x : A_{(k+1)}. B_{(k+1)}$
Atomic types	$P_{(k+1)}$	$::=$	$a \cdot S_{(k+1)}$
Canonical terms	$M_{(k+1)}$	$::=$	$\lambda x. M_{(k+1)} \mid R_{(k+1)}$
Neutral terms	$R_{(k+1)}$	$::=$	$x \cdot T_{(k+1)} \mid c \cdot S_{(k+1)}$
Continuing Spines	$T_{(k+1)}$	$::=$	$() \mid M_{(k+1)}; T_{(k+1)}$
Suspended Spines	$S_{(k+1)}$	$::=$	$() \mid M_{(k)}; S_{(k+1)}$

There is a correspondence between the structure of canonical dependent types and the simple types. We use the erasure operation  $(-)^o$  (super script  $o$ ) to map an observable canonical type to a simple type  $\tau$ . For example,  $(\Pi x : a. a_2 \cdot (x))^o = * \rightarrow *$ . We also define the hereditary substitution of a canonical term into kinds, canonical and atomic types, and contexts. They are defined by the following judgments.

$A_{(k+1)}^o = \tau$	Observable type $A$ erases to $\tau$
$[N_{(k)}/x]^\tau K_{(k+1)} =_{(k+1)} K'_{(k+1)}$	Hereditary substitution in types
$[N_{(k)}/x]^\tau A_{(k+1)} =_{(k+1)} A'_{(k+1)}$	Hereditary substitution in canonical type
$[N_{(k)}/x]^\tau P_{(k+1)} =_{(k+1)} P'_{(k+1)}$	Hereditary substitution in atomic type
$[N_{(k)}/x]^\tau \Gamma_{(k+1)} =_{(k+1)} \Gamma'_{(k+1)}$	Hereditary substitution in contexts

The operation of substitution is also defined by lexicographic induction on  $\tau$ ,  $k$  and the structure of the expression on the right-hand side of  $\tau$  ( $K, A, P, \Gamma$ ).

$A_{(k+1)}^o = \tau$
$((\Pi x : A_2. A_1)_{(k+1)})^o = ((A_2)_{(k+1)}^o) \rightarrow ((A_1)_{(k+1)}^o)$
$(P_{(k+1)})^o = *$
$[N_{(k)}/x]^\tau K_{(k+1)} =_{(k+1)} K'_{(k+1)}$
$[N_{(k)}/x]^\tau \text{type} =_{(k+1)} \text{type}$
$[N_{(k)}/x]^\tau \text{cotype} =_{(k+1)} \text{cotype}$
$[N_{(k)}/x]^\tau \Pi y : A_{(k+1)}. K_{(k+1)} = \Pi y : A'_{(k+1)}. K'_{(k+1)}$ if $[N_{(k)}/x]^\tau A_{(k+1)} =_{(k+1)} A'_{(k+1)}$ and $[N_{(k)}/x]^\tau K_{(k+1)} =_{(k+1)} K'_{(k+1)}$
$[N_{(k)}/x]^\tau A_{(k+1)} =_{(k+1)} A'_{(k+1)}$
$[N_{(k)}/x]^\tau P_{(k+1)} = P'_{(k+1)}$ if $[N_{(k)}/x]^\tau P_{(k+1)} =_{(k+1)} P'_{(k+1)}$
$[N_{(k)}/x]^\tau \Pi y : B_{(k+1)}. A_{(k+1)} =_{(k+1)} \Pi y : B'_{(k+1)}. A'_{(k+1)}$ if $[N_{(k)}/x]^\tau B_{(k+1)} =_{(k+1)} B'_{(k+1)}$ and $[N_{(k)}/x]^\tau A_{(k+1)} =_{(k+1)} A'_{(k+1)}$
$[N_{(k)}/x]^\tau P_{(k+1)} =_{(k+1)} P'_{(k+1)}$
$[N_{(k)}/x]^\tau (a \cdot S_{(k+1)}) =_{(k+1)} a \cdot S'_{(k+1)}$ if $[N_{(k)}/x]^\tau S_{(k+1)} =_{(k+1)} S'_{(k+1)}$
$[N_{(k)}/x]^\tau \Gamma_{(k+1)} =_{(k+1)} \Gamma'_{(k+1)}$
$[N_{(k)}/x]^\tau \cdot =_{(k+1)} \cdot$
$[N_{(k)}/x]^\tau (\Gamma_{(k+1)}, y : A_{(k+1)}) =_{(k+1)} \Gamma'_{(k+1)}, y : A_{(k+1)}$ if $[N_{(k)}/x]^\tau \Gamma_{(k+1)} =_{(k+1)} \Gamma'_{(k+1)}$ and $[N_{(k)}/x]^\tau A_{(k+1)} =_{(k+1)} A'_{(k+1)}$

**THEOREM 5.1 (HEREDITARY SUBSTITUTION RESPECTS OBSERVATION DEPTH).** *If both  $N_{(k)}$  is a term of observation depth  $k$ , and  $E_{(k+1)}$  ( $E$  is  $K, A, P$ ) is a kind/type of depth  $k + 1$ , then for all  $\tau$ ,  $[N_{(k)}/x]^\tau E_{(k+1)}$  is of productive depth  $k + 1$  if defined.*

**PROOF.** Straightforward lexicographic induction on  $\tau$ ,  $k$ , and the structure of  $E_{(k+1)}$ . □

## 5.2 Type Checking Rules

We simultaneously define the following type checking judgments, by induction on  $k$  and the structure of the subject expression. All judgments except  $\Sigma_{(k)} \text{ sig}$  presuppose  $\Sigma_{(k)} \text{ sig}$ . All judgments with  $\Gamma_{(k)}$  present presuppose  $\vdash_{\Sigma_{(k)}} \Gamma_{(k)} \text{ ctx}$ .

$\Sigma_{(k)} \text{ sig}$	Signature $\Sigma_{(k)}$ is valid
$\vdash_{\Sigma_{(k)}} \Gamma_{(k)} \text{ ctx}$	Context $\Gamma_{(k)}$ is well-formed
$\Gamma_{(k)} \vdash_{\Sigma_{(k)}} K_{(k)} \Leftarrow \text{kind}$	Kind $K_{(k)}$ is a valid kind
$\Gamma_{(k)} \vdash_{\Sigma_{(k)}} A_{(k)} \Leftarrow (\text{co})\text{type}$	Type $A_{(k)}$ is a canonical (co)type
$\Gamma_{(k)} \vdash_{\Sigma_{(k)}} P_{(k)} \Rightarrow K_{(k)}$	Atomic type $P_{(k)}$ synthesizes kind $K_{(k)}$
$\Gamma_{(k)} \vdash_{\Sigma_{(k)}} S_{(k)} \triangleright K_{(k)} \Rightarrow K'_{(k)}$	Suspended Spine $S_{(k)}$ applied to kind $K_{(k)}$ produces kind $K'_{(k)}$
$\Gamma_{(k)} \vdash_{\Sigma_{(k)}} M_{(k)} \Leftarrow A_{(k)}$	Term $M_{(k)}$ checks against type $A_{(k)}$
$\Gamma_{(k)} \vdash_{\Sigma_{(k)}} R_{(k)} \Rightarrow P_{(k)}$	Neutral term $R_{(k)}$ synthesizes type $P_{(k)}$
$\Gamma_{(k)} \vdash_{\Sigma_{(k)}} T_{(k)} \triangleright A_{(k)} \Rightarrow P_{(k)}$	Continuing Spine $T_{(k)}$ applied to type $A_{(k)}$ produces type $P_{(k)}$
$\Gamma_{(k)} \vdash_{\Sigma_{(k)}} S_{(k)} \triangleright A_{(k)} \Rightarrow P_{(k)}$	Suspended Spine $S_{(k)}$ applied to type $A_{(k)}$ produces type $P_{(k)}$

$\vdash \Sigma_{(k)} \text{ sig}$

$$\frac{}{\vdash \Sigma_{(0)} \text{ sig}} \quad \frac{}{\vdash \cdot \text{ sig}} \quad \frac{\vdash \Sigma_{(k+1)} \text{ sig} \quad \cdot \vdash K_{(k+1)} \Leftarrow \text{kind}}{\vdash \Sigma_{(k+1)}, a : K_{(k+1)} \text{ sig}}$$

$$\frac{\vdash \Sigma_{(k+1)} \text{ sig} \quad \cdot \vdash A_{(k+1)} \Leftarrow (\text{co})\text{type}}{\vdash \Sigma_{(k+1)}, c : A_{(k+1)} \text{ sig}}$$

$\vdash_{\Sigma_{(k)}} \Gamma_{(k)} \text{ ctx}$

$$\frac{}{\vdash_{\Sigma_{(0)}} \Gamma_{(0)} \text{ ctx}} \quad \frac{}{\vdash_{\Sigma_{(k+1)}} \cdot \text{ ctx}} \quad \frac{\vdash_{\Sigma_{(k+1)}} \Gamma_{(k+1)} \text{ ctx} \quad \Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} A_{(k+1)} \Leftarrow (\text{co})\text{type}}{\vdash \Gamma_{(k+1)}, x : A_{(k+1)} \text{ ctx}}$$

$\Gamma_{(k)} \vdash_{\Sigma_{(k)}} K_{(k)} \Leftarrow \text{kind}$

$$\frac{}{\Gamma_{(0)} \vdash_{\Sigma_{(0)}} K_{(0)} \Leftarrow \text{kind}} \quad \frac{}{\Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} \text{ type} \Leftarrow \text{kind}} \quad \frac{}{\Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} \text{ cotype} \Leftarrow \text{kind}}$$

$$\frac{\Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} A_{(k+1)} \Leftarrow (\text{co})\text{type} \quad \Gamma_{(k+1)}, x : A_{(k+1)} \vdash_{\Sigma_{(k+1)}} K_{(k+1)} \Leftarrow \text{kind}}{\Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} \Pi x : A_{(k+1)}. K_{(k+1)} \Leftarrow \text{kind}}$$

$\Gamma_{(k)} \vdash_{\Sigma_{(k)}} A_{(k)} \Leftarrow (\text{co})\text{type}$

$$\frac{}{\vdash_{\Sigma_{(0)}} A_{(0)} \Leftarrow (\text{co})\text{type}}$$

$$\frac{\Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} B_{(k+1)} \Leftarrow (\text{co})\text{type} \quad \Gamma_{(k+1)}, x : B_{(k+1)} \vdash_{\Sigma_{(k+1)}} A_{(k+1)} \Leftarrow (\text{co})\text{type}}{\Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} \Pi x : B_{(k+1)}. A_{(k+1)} \Leftarrow (\text{co})\text{type}}$$

$$\frac{\Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} P_{(k+1)} \Rightarrow K_{(k+1)} \quad K_{(k+1)} = \text{type/cotype}}{\Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} P_{(k+1)} \Leftarrow (\text{co})\text{type}}$$

$$\boxed{\Gamma_{(k)} \vdash_{\Sigma_{(k)}} P_{(k)} \Rightarrow K_{(k)}}$$

$$\frac{\Gamma_{(0)} \vdash_{\Sigma_{(0)}} P_{(0)} \Leftarrow K_{(0)}}{\Gamma_{(k)} \vdash_{\Sigma_{(k)}} P_{(k)} \Leftarrow K_{(k)}} \quad \frac{a : K_{(k+1)} \in \Sigma_{(k+1)} \quad \Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} S_{(k+1)} \triangleright K_{(k+1)} \Rightarrow K'_{(k+1)}}{\Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} a \cdot S_{(k+1)} \Rightarrow K'_{(k+1)}}$$

$$\boxed{\Gamma_{(k)} \vdash_{\Sigma_{(k)}} S_{(k)} \triangleright K_{(k)} \Rightarrow K'_{(k)}}$$

$$\frac{\Gamma_{(0)} \vdash_{\Sigma_{(0)}} S_{(0)} \triangleright K_{(0)} \Rightarrow K'_{(0)}}{\Gamma_{(k)} \vdash_{\Sigma_{(k)}} S_{(k)} \triangleright K_{(k)} \Rightarrow K'_{(k)}} \quad \frac{\Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} () \triangleright K_{(k+1)} \Rightarrow K_{(k+1)}}{\Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} () \triangleright K_{(k+1)} \Rightarrow K_{(k+1)}}$$

$$\frac{\Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} M_{(k+1)} \Leftarrow A_{(k+1)} \quad [M_{(k)}/x]^{(A_{(k+1)})^\circ} K_{(k+1)} =_{(k+1)} K'_{(k+1)} \quad \Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} S_{(k+1)} \triangleright K'_{(k+1)} \Rightarrow K''_{(k+1)}}{\Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} M_{(k)}; S_{(k+1)} \triangleright \Pi x : A_{(k+1)} \cdot K_{(k+1)} \Rightarrow K''_{(k+1)}}$$

$$\boxed{\Gamma_{(k)} \vdash_{\Sigma_{(k)}} M_{(k)} \Leftarrow A_{(k)}}$$

$$\frac{\Gamma_{(0)} \vdash_{\Sigma_{(0)}} M_{(0)} \Leftarrow A_{(0)}}{\Gamma_{(k)} \vdash_{\Sigma_{(k)}} M_{(k)} \Leftarrow A_{(k)}} \quad \frac{\Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} R_{(k+1)} \Rightarrow P'_{(k+1)} \quad P'_{(k+1)} =_{(k+1)} P_{(k+1)}}{\Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} R_{(k+1)} \Leftarrow P_{(k+1)}}$$

$$\frac{\Gamma_{(k+1)}, x : B_{(k+1)} \vdash_{\Sigma_{(k+1)}} M_{(k+1)} \Leftarrow A_{(k+1)}}{\Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} \lambda x. M_{(k+1)} \Leftarrow \Pi x : B_{(k+1)}. A_{(k+1)}}$$

$$\boxed{\Gamma_{(k)} \vdash_{\Sigma_{(k)}} R_{(k)} \Rightarrow P_{(k)}}$$

$$\frac{\Gamma_{(0)} \vdash_{\Sigma_{(0)}} R_{(0)} \Rightarrow P_{(0)}}{\Gamma_{(k)} \vdash_{\Sigma_{(k)}} R_{(k)} \Rightarrow P_{(k)}} \quad \frac{x : A_{(k+1)} \in \Gamma_{(k+1)} \quad \Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} T_{(k+1)} \triangleright A_{(k+1)} \Rightarrow P_{(k+1)}}{\Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} x \cdot T_{(k+1)} \Rightarrow P_{(k+1)}}$$

$$\frac{c : A_{(k+1)} \in \Sigma \quad \Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} S_{(k+1)} \triangleright A_{(k+1)} \Rightarrow P_{(k+1)}}{\Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} c \cdot S_{(k+1)} \Rightarrow P_{(k+1)}}$$

$$\boxed{\Gamma_{(k)} \vdash_{\Sigma_{(k)}} T_{(k)} \triangleright A_{(k)} \Rightarrow P_{(k)}}$$

$$\frac{\Gamma_{(0)} \vdash_{\Sigma_{(0)}} T_{(0)} \triangleright A_{(0)} \Rightarrow P_{(0)}}{\Gamma_{(k)} \vdash_{\Sigma_{(k)}} T_{(k)} \triangleright A_{(k)} \Rightarrow P_{(k)}} \quad \frac{\Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} () \triangleright P_{(k+1)} \Rightarrow P_{(k+1)}}{\Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} () \triangleright P_{(k+1)} \Rightarrow P_{(k+1)}}$$

$$\frac{\Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} M_{(k+1)} \Leftarrow B_{(k+1)} \quad [M_{(k)}/x]^{B^\circ} A_{(k+1)} =_{(k+1)} A'_{(k+1)} \quad \Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} T_{(k+1)} \triangleright A'_{(k+1)} \Rightarrow P_{(k+1)}}{\Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} M_{(k+1)}; T_{(k+1)} \triangleright \Pi x : B_{(k+1)}. A_{(k+1)} \Rightarrow P_{(k+1)}} (*)$$

$$\boxed{\Gamma_{(k)} \vdash_{\Sigma_{(k)}} S_{(k)} \triangleright A_{(k)} \Rightarrow P_{(k)}}$$

$$\frac{\Gamma_{(0)} \vdash_{\Sigma_{(0)}} S_{(0)} \triangleright A_{(0)} \Rightarrow P_{(0)}}{\Gamma_{(k)} \vdash_{\Sigma_{(k)}} S_{(k)} \triangleright A_{(k)} \Rightarrow P_{(k)}} \quad \frac{\Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} () \triangleright P_{(k+1)} \Rightarrow P_{(k+1)}}{\Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} () \triangleright P_{(k+1)} \Rightarrow P_{(k+1)}}$$

$$\frac{\Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} M_{(k+1)} \Leftarrow B_{(k+1)} \quad [M_{(k)}/x]^{B^\circ} A_{(k+1)} =_{(k+1)} A'_{(k+1)} \quad \Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} S_{(k+1)} \triangleright A'_{(k+1)} \Rightarrow P_{(k+1)}}{\Gamma_{(k+1)} \vdash_{\Sigma_{(k+1)}} M_{(k)}; S_{(k+1)} \triangleright \Pi x : B_{(k+1)}. A_{(k+1)} \Rightarrow P_{(k+1)}}$$

We state and prove substitution lemmas for CoLF<sup>ω</sup>.

**THEOREM 5.2 (SUBSTITUTION).** *Given  $\Sigma$ , and depth  $k$ , we have*

(1) *Substitution in canonical terms:*

*If  $\Gamma_{(k+1)} \vdash N_{(k+1)} \Leftarrow A_{(k+1)}$ , and  $\Gamma_{(k+1)}, x : A_{(k+1)}, \Gamma'_{(k+1)} \vdash M_{(k+1)} \Leftarrow B_{(k+1)}$ ,  
then  $\Gamma_{(k+1)}, [N_{(k)}/x] \Gamma'_{(k+1)} \vdash [N_{(k+1)}/x] M_{(k+1)} \Leftarrow [N_{(k)}/x] B_{(k+1)}$ .*

(2) *Substitution in canonical types:*

*If  $\Gamma_{(k+1)} \vdash N_{(k+1)} \Leftarrow_{(k+1)} A$ , and  $\Gamma_{(k+1)}, x : A_{(k+1)}, \Gamma'_{(k+1)} \vdash B_{(k+1)} \Leftarrow (\text{co})\text{type}$ , then  
 $\Gamma_{(k+1)}, [N_{(k)}/x] \Gamma'_{(k+1)} \vdash [N_{(k)}/x] B_{(k+1)} \Leftarrow (\text{co})\text{type}$ .*

**PROOF.** Induction on  $\tau$ ,  $k$ , and the structure of  $M$  or  $B$ . This proof is similar to the ones in Harper and Licata [2007]; Watkins et al. [2002].  $\square$

### 5.3 Validity of Infinitary Terms

With signature  $\Sigma_{(\omega)}$ , we say that a type  $A$  is inductive if  $A = \Pi x_1 \dots \Pi x_n : A_n.a \cdot S$  and  $a : \Pi y_1 \dots \Pi y_m : B_m$ . type, and a type  $A$  coinductive if  $A = \Pi x_1 \dots \Pi x_n : A_n.a \cdot S$  and  $a : \Pi y_1 \dots \Pi y_m : B_m$ . cotype. A constructor  $c$  is inductive if  $c : A \in \Sigma$  and  $A$  is inductive, and  $c$  is coinductive if  $c : A \in \Sigma$  and  $A$  is coinductive. As with CoLF [Chen and Pfenning 2023], a priority is assigned to each type: types and type families whose kinds are declared later in the signature have higher priority than types whose kinds are declared earlier. Term constructors inherit priorities from their types.

Intuitively, an infinitary term  $M$  is valid if for all infinitary traces  $T$  of  $M$ , some coinductive constructor occurs infinitely often along the trace  $T$ . In particular, a term in which there is an infinite stack of inductive constructors is not valid.

Recall that a trace is a sequence of constructor constants or variables, whose length is possibly infinite. A term  $M$  is *trace-valid* if for all infinite traces  $T$  of  $M$ , there is a coinductive constructor that occurs infinitely often along  $T$ , and that coinductive constructor has a higher priority than any other constructor that occurs infinitely often along  $T$ . In other words, for any infinite trace  $T$  of  $M$ , the highest priority constructor that occurs infinitely often along  $T$  must be coinductive.

**THEOREM 5.3 (VALIDITY ENTAILS PRODUCTIVITY).** *If  $M$  is trace-valid, then  $M$  is productive.*

**PROOF.** Along any infinite trace  $T$  through  $M$ , there exist infinitely many coinductive constructors along  $T$  by trace-validity, and then it is productive by the presence of constructors.  $\square$

We note that hereditary substitution preserves the validity of terms.

**THEOREM 5.4 (TRACE-VALIDITY CLOSED UNDER HEREDITARY SUBSTITUTION).** *If both  $N$  and  $M$  are trace-valid, then  $[N/x]^r M$  is trace-valid if defined.*

**PROOF.** An analysis of the definition of hereditary substitution reveals that any infinite trace of  $[N/x]^r M$  is a zipping of a trace  $T_2$  of  $N$  and  $T_1$  of  $M$ , at least one of which is infinite. Then since both  $N$  and  $M$  are productive, the highest priority constructor that occurs infinitely often along  $T_2$  or  $T_1$  or both (depending on which ones are infinite) will be coinductive. Thus, the infinite trace formed as a result of zipping will have an infinitely occurring coinductive constructor of the highest priority.  $\square$

## 6 INTERPRETATION OF FINITARY SIGNATURES

We now present a finitary signature  $\Sigma$  in the style of CoLF [Chen and Pfenning 2023] that can be interpreted into a  $\text{CoLF}^\omega$  signature  $\Sigma_{(\omega)}$ . Type checking in  $\text{CoLF}^\omega$  is undecidable up to depth  $\omega$ , but is decidable for any finite depth  $k < \omega$ . Moreover, if the recursive definitions in the finitary signature satisfy the prepattern restriction of CoLF, that all recursion constants are applied to

bound variables, then we have decidable type checking up to depth  $\omega$ . The reason is that if the terms that appear in the signature are all rational, then the signature may be regarded as a CoLF signature for the purpose of type checking, while semantically remains to be interpreted in  $\text{CoLF}^\omega$ .

Signatures	$\Sigma$	$::= \cdot \mid \Sigma, a : K \mid \Sigma, c : A \mid \Sigma, r : A = M$
Contexts	$\Gamma$	$::= \cdot \mid \Gamma, x : A$
Kinds	$K$	$::= \text{type} \mid \text{cotype} \mid \Pi x : A. K$
Canonical types	$A, B$	$::= P \mid \Pi x : B. A$
Atomic types	$P$	$::= a \cdot S$
Canonical terms	$M$	$::= R \mid \lambda x. M$
Neutral terms	$R$	$::= x \cdot S \mid c \cdot S \mid r \cdot S$
Spines	$S$	$::= () \mid M; S$

The infinitary terms are given by terms computed by a fixed point definition. To ensure that recursive definitions are still contractive and productive, we require the head of  $M$  in a definition  $r : A = M$  to be a constant  $c$ .<sup>3</sup> We define the expansion of a finitary CoLF signature  $\Sigma$  into an infinitary  $\text{CoLF}^\omega$  signature of depth  $k$  by the function  $\text{exp}_{(k)}$ , and similarly for all syntax categories. The expansion functions are implicitly parametrized by a signature  $\Sigma$ , providing the recursive definitions. We have two choices when expanding a CoLF spine  $S$ , a suspended spine  $S_{(k)}$  or a continuing spine  $T_{(k)}$ . We use  $\text{exp}_{(k)}^S$  for expansion into suspended spines and  $\text{exp}_{(k)}^T$  for expansion into continuing spines. The definition of the expansion is as follows.

$\text{exp}_{(k)}(\Sigma) =_{(k)} \Sigma_{(k)}$	
$\text{exp}_{(0)}(\Sigma)$	$=_{(0)} \Sigma_{(0)}$
$\text{exp}_{(k+1)}(\Sigma, a : K)$	$=_{(k+1)} \text{exp}_{(k+1)}(\Sigma), a : K_{(k+1)}$
$\text{exp}_{(k+1)}(\Sigma, c : A)$	$=_{(k+1)} \text{exp}_{(k+1)}(\Sigma), c : A_{(k+1)}$
$\text{exp}_{(k+1)}(\Sigma, r : A = M)$	$=_{(k+1)} \text{exp}_{(k+1)}(\Sigma)$
$\text{exp}_{(k)}(\Gamma) =_{(k)} \Gamma_{(k)}$	
$\text{exp}_{(0)}(\Gamma) =_{(0)} \Gamma_{(0)}$	
$\text{exp}_{(k+1)}(\Gamma, x : A)$	$=_{(k+1)} \text{exp}_{(k+1)}(\Gamma), x : A_{(k+1)}$
$\text{exp}_{(k)}(K) =_{(k)} K_{(k)}$	
$\text{exp}_{(0)}(K) =_{(0)} K_{(0)}$	
$\text{exp}_{(k+1)}(\text{type})$	$=_{(k+1)} \text{type}$
$\text{exp}_{(k+1)}(\text{cotype})$	$=_{(k+1)} \text{cotype}$
$\text{exp}_{(k+1)}(\Pi x : A. K)$	$=_{(k+1)} \Pi x : \text{exp}_{(k+1)}(A). \text{exp}_{(k+1)}(K)$
$\text{exp}_{(k)}(A) =_{(k)} A_{(k)}$	
$\text{exp}_{(0)}(A) =_{(0)} A_{(0)}$	
$\text{exp}_{(k+1)}(P)$	$=_{(k+1)} \text{exp}_{(k+1)}(P)$
$\text{exp}_{(k+1)}(\Pi x : B. A)$	$=_{(k+1)} \Pi x : \text{exp}_{(k+1)}(B). \text{exp}_{(k+1)}(A)$
$\text{exp}_{(k)}(P) =_{(k)} P_{(k)}$	
$\text{exp}_{(0)}(P) =_{(0)} P_{(0)}$	
$\text{exp}_{(k+1)}(a \cdot S)$	$=_{(k+1)} a \cdot \text{exp}_{(k+1)}^S(S)$

<sup>3</sup>This turns out to be more restrictive than CoLF, because we would like to rule out definitions, e.g.  $r = \lambda x.x (r x)$ , which are not productive. CoLF signatures allow nonproductive terms because substitutions for recursive terms are restricted to variable renaming only.

$$\begin{array}{l}
\boxed{\exp_{(k)}(M) =_{(k)} M_{(k)}} \\
\exp_{(0)}(M) =_{(0)} M_{(0)} \\
\exp_{(k+1)}(\lambda x. M) =_{(k+1)} \lambda x. \exp_{(k+1)}(M) \\
\exp_{(k+1)}(R) =_{(k+1)} \exp_{(k+1)}(R) \\
\boxed{\exp_{(k)}(R) =_{(k)} R_{(k)}} \\
\exp_{(0)}(R) =_{(0)} R_{(0)} \\
\exp_{(k+1)}(c \cdot S) =_{(k+1)} c \cdot (\exp_{(k)}^S(S)) \\
\exp_{(k+1)}(x \cdot S) =_{(k+1)} x \cdot (\exp_{(k)}^T(S)) \\
\exp_{(k+1)}(r \cdot S) =_{(k+1)} \exp_{(k+1)}^T(S) \triangleright^\tau \exp_{(k+1)}(M) \\
\text{if } r : A = M \in \Sigma \text{ and } \tau = (\exp_{(k+1)}(A))^o \\
\boxed{\exp_{(k)}^T(S) =_{(k)} T_{(k)}} \\
\exp_{(0)}^T(S) =_{(0)} T_{(0)} \\
\exp_{(k+1)}^T() =_{(k+1)} () \\
\exp_{(k+1)}^T(M; S) =_{(k+1)} (\exp_{(k+1)}(M)); (\exp_{(k+1)}^T(S)) \\
\boxed{\exp_{(k)}^S(S) =_{(k)} S_{(k)}} \\
\exp_{(0)}^S(S) =_{(0)} S_{(0)} \\
\exp_{(k+1)}^S() =_{(k+1)} () \\
\exp_{(k+1)}^S(M; S) =_{(k+1)} (\exp_{(k)}(M)); (\exp_{(k+1)}^S(S))
\end{array}$$

**THEOREM 6.1 (SEMI DECIDABILITY OF TYPE CHECKING).** *Given a finitary signature  $\Sigma$ , it is decidable whether  $\vdash_{(k)} \exp_{(k)}(\Sigma)$  sig for any  $k$ . And similarly for other type checking judgments.*

**PROOF.** Given a finitary signature, its depth  $k$  expansion can be computed. The type checking judgments are defined by induction on the observation depth and are thus decidable.  $\square$

## 6.1 Infinitary Canonical Forms

Given a finitary signature  $\Sigma$ , let  $\Sigma_{(\omega)}$  be the expansion  $\exp_{(\omega)}(\Sigma)$ . For a type  $a$  : type or  $a$  : cotype in  $\Sigma$ , the possibly infinitary canonical terms  $M_{(\omega)}$  for the type  $a$  in a context  $\Gamma_{(\omega)}$  are such that  $\Gamma_{(\omega)} \vdash_{\Sigma_{(\omega)}} M_{(\omega)} \Leftarrow a$ , and  $M_{(\omega)}$  is valid.  $\Gamma_{(\omega)} \vdash_{\Sigma_{(\omega)}} M_{(\omega)} \Leftarrow a$  means that for all observation depth  $k$ , the partial observation of  $M_{(\omega)}$  at depth  $k$ ,  $M_{(k)}$ , satisfies  $\Gamma_{(k)} \vdash_{\Sigma_{(k)}} M_{(k)} \Leftarrow a$ .

The above definition may generalize to type families indexed by terms. That is, for a type family  $a : \Pi x_1 : A_1. \dots, \Pi x_n : A_n$ . type or  $a : \Pi x_1 : A_1. \dots, \Pi x_n : A_n$ . cotype, the possibly infinitary canonical terms  $M_{(\omega)}$  for the type  $a$  indexed by a spine of possibly infinitary terms,  $a \cdot ((M_1)_{(\omega)}, \dots, (M_n)_{(\omega)})$  in a context  $\Gamma_{(\omega)}$  are such that  $\Gamma_{(\omega)} \vdash_{\Sigma_{(\omega)}} M_{(\omega)} \Leftarrow a \cdot ((M_1)_{(\omega)}, \dots, (M_n)_{(\omega)})$ , and  $M_{(\omega)}$  is valid. That is, for all observation depth  $k > 0$ , the partial observation of  $M_{(\omega)}$  at depth  $k$ ,  $M_{(k)}$ , satisfies  $\Gamma_{(k)} \vdash_{\Sigma_{(k)}} M_{(k)} \Leftarrow a \cdot ((M_1)_{(k-1)}, \dots, (M_n)_{(k-1)})$ , and  $M_{(\omega)}$  needs to satisfy the global validity condition. In both cases, we may omit the phrase “in a context  $\Gamma_{(\omega)}$ ” and just say  $M_{(\omega)}$  is a canonical term for a type  $a$  with a possible spine, if  $\Gamma_{(\omega)}$  is an empty context.

The reader may verify the signatures presented in Section 2 provide adequate encodings of infinitary objects. As with Section 2, we use the Twelf’s concrete syntax [Pfenning and Schürmann 1999] in the presentation of finitary signatures, where  $\Pi$ -abstractions are written using curly braces and  $\lambda$ -abstractions are written using square brackets. Implicit  $\Pi$ -abstractions are created for capitalized types.



## 7 ENCODING PRODUCTIVE BÖHM TREES

We present an encoding of  $\perp$ -free Böhm trees (not necessarily productive) in  $\text{CoLF}^\omega$ . Then we present two ways to encode productivity.

Below is the signature ( $\Sigma_4$ ) for encoding  $\perp$ -free Böhm trees. `ctm` stands for canonical terms, `ntm` stands for neutral terms, `vars` stands for a distinguished set of variables, and `consts` stands for a distinguished set of constants. Both `ctm` and `spine` are finite because we don't want terms consisting of infinitely many consecutive  $\lambda$ -abstractions or spines of infinitary length.

```

%% Signature 4:
ctm : type.
spine : type.
ntm : cotype.
vars : type.
consts : type.

lam : (vars -> ctm) -> ctm.
base : ntm -> ctm.
varntm : vars -> spine -> ntm.
constntm : consts -> spine -> ntm.
snil : spine.
scons : ctm -> spine -> spine.

```

The term  $I = \lambda x. x$  may be encoded as `lam ([x] base (varntm x snil))`. The term  $Z = \lambda x. x \cdot (\lambda y. y \cdot (\lambda z. z \cdot (\dots)))$ , not productive and not typeable, can be encoded as the  $\Sigma_4$ -typeable term `lam ([x] base (varntm x (scons (...) snil)))` of type `ctm`. The  $\dots$  will be another encoding of the term  $Z$ , continuing indefinitely. Let the encoding of  $Z$  subsequently be referred to as `tmZ`. `tmZ` is a valid term because of the coinductive constructor `varntm` infinitely occurring along the only one infinite trace through `tmZ`. Note that `tmZ` also has infinitely many finite traces such as `lam`, `base`, `varntm`, `scons`, `snil`, which validity does not concern.

**THEOREM 7.1 (ADEQUACY OF ENCODING).** *There is a bijection between the  $\perp$ -free Böhm trees and the infinitary canonical terms of type `ctm`.*

**PROOF.** We impose an observation structure on  $\perp$ -free Böhm trees, such that a single observation may reveal a  $\lambda$ -abstraction, a head-spine form, or the next element in a spine. Then, we can establish bijective correspondences between  $\perp$ -free Böhm trees and their encodings by induction on the observation depth.

The validity condition on the canonical terms of type `ctm` corresponds to the exact requirement of  $\perp$ -free Böhm trees, that there are no infinite chains of  $\lambda$ -abstractions or applications.  $\square$

### 7.1 Encoding the Productivity Condition

We present two different ways we can incorporate productivity conditions, an external approach and an internal approach. Both approaches encode object-level productivity as framework-level validity. The external approach encodes the productivity condition externally as a predicate on terms, and no modification to the previous signature is required. We define new predicates beginning with `prod` on all three syntactic classes and clauses for them below ( $\Sigma_5$ ). The suffix `c` means canonical terms, `n` means neutral terms, `t` means continuing spines, and `s` means suspended spine. Constants are followed by suspended spines, which are the only progress points.

```

%% Signature 5:
prodc : ctm -> type.
prodn : ntm -> type.
prodt : spine -> type.
prods : spine -> cotype.

```

```

p1 : ({x} prodc (M x)) -> prodc (lam M).
p2 : prodn R -> prodc (base R).
p3 : prodt S -> prodn (varntm V S).
p4 : prods S -> prodn (constntm C S).
p5 : prodt snil.
p6 : prodc M -> prodt S -> prodt (scons M S).
p7 : prods snil.
p8 : prodc M -> prods S -> prods (scons M S).

```

The rules are defined to be syntax-directed so given any term  $t$  of type  $\text{ctm}$ , a search for the derivation  $\text{prodc } t$  (i.e. a term of type  $\text{prodc } t$ ) will deterministically pick rules between  $p1$  and  $p8$  corresponding to the syntactic structure. For example, let the term  $\text{tmI}$  be the encoding of  $I = \lambda x. x$ , a search for a term of the type  $\text{prodc } \text{tmI}$  will be  $p1 \ ([x] \ p2 \ (p3 \ p5))$ , which is a valid term because it is finite.

Within the signature  $\Sigma_5$ , observe that the only coinductive type is  $\text{prods}$ , and the only coinductive constructor constructors are  $p7$  and  $p8$  because they are constructors for the coinductive type family  $\text{prods}$ . Given a nonproductive term  $t$ , there cannot be a valid term of type  $\text{prodc } t$ . Every infinite trace of  $t$  will correspond to some infinite trace in a term of type  $\text{prodc } t$ , and for every constant along that trace, each element in its spine will correspond to a coinductive constructor  $p8$ . Since  $t$  is not productive, there exists an infinite trace with only finitely many constants, and then the corresponding infinite trace of  $\text{prodc } t$  will contain only finitely many  $p8$ 's, and thus is not a valid term. For example, consider the supposed construction of the term  $\text{pTmZ}$  of type  $\text{prodc } \text{tmZ}$  where  $\text{tmZ}$  encodes  $Z$ , since there are no constants in  $Z$ , the term  $\text{pTmZ}$  does not involve  $p7$  or  $p8$ . It is infinite because it follows the syntax of  $\text{tmZ}$ , and therefore it cannot be valid. In summary, the predicate beginning with  $\text{prod}$  encodes the productivity of the underlying term.

**THEOREM 7.2 (ADEQUACY OF ENCODING).** *Given a  $\perp$ -free Böhm tree  $M$ , and its infinite encoding  $\ulcorner M \urcorner$ , the  $M$  is valid if and only if the type  $\text{prodc } \ulcorner M \urcorner$  is inhabited.*

**PROOF.** Since  $\text{prodc}$  is defined in a syntax-directed way, given a term  $M$ , there exists a unique term  $M'_{(\omega)} = \ulcorner M \urcorner$  such that  $\vdash_{\Sigma(\omega)} M'_{(\omega)} \Leftarrow \text{prodc } M_{(\omega)}$ . Then, it suffices to show that the validity condition on  $M'_{(\omega)}$  encodes the productivity condition. Any infinite trace of  $M'_{(\omega)}$  corresponds to an infinite trace of  $M$ , and the productivity condition on the infinite traces of  $M$  exactly correspond to the validity condition of  $M'_{(\omega)}$ . □

Let us now turn to the internal approach. In the internal approach, instead of having a separate predicate encoding the validity, we postulate that only productive infinite terms are well-typed terms, and nonproductive are invalid by definition. The signature ( $\Sigma_6$ ) below shows the internal approach. The progress points are modeled through progress canonical terms ( $\text{p\_ctm}$ ), which appear only in suspended spines ( $\text{s\_spine}$ ) following constants.

```

% Signature 6:
ctm : type.
ntm : type.
t_spine : type.
s_spine : cotype.
p_ctm : cotype.
vars : type.
consts : type.

lam : (vars -> ctm) -> ctm.
base : ntm -> ctm.
varntm : vars -> t_spine -> ntm.
constntm : consts -> s_spine -> ntm.
tnil : t_spine.
tcons : ctm -> t_spine -> t_spine.
snil : s_spine.
scons : p_ctm -> s_spine -> s_spine.
progress : ctm -> p_ctm.

```

In  $\Sigma_6$ , the only coinductive constructor is `progress`, which is used to guard the terms in suspended spines following constants. We can see that the productivity of infinitary  $\lambda$ -terms is encoded as the validity of  $\text{CoLF}^\omega$ . The encoding of  $I = \lambda x. x$  will be `lam ([x] base (varntm x tnil))`, which is valid because it is finite. The encoding of  $Z = \lambda x. x \cdot (\lambda y. y \cdot (\lambda z. z \cdot (\dots)))$  is `lam ([x] base (varntm x (tcons (... tnil))))`, where  $\dots$  denotes the same encoding of  $Z$ . This term is not valid because the encoding does not contain the only coinductive constructor `progress` and there exists an infinite trace through the term. If we fix  $c$  as a constant, `c : const`, the term  $Y = \lambda x. c \cdot (x \cdot (c \cdot (x \cdot (\dots))))$ , is encoded as `lam ([x] base (constntm c (scons (progress (base (varntm x (tcons (... tnil)))) snil))))`. This encoding is valid because of the occurrences of `progress` along the infinite trace through the term

**THEOREM 7.3 (ADEQUACY OF ENCODING).** *There is a bijection between the productive  $\perp$ -free Böhm trees  $\lambda$ -terms and the infinitary canonical terms of type `ctm`.*

**PROOF.** As with the previous proof, we stratify the  $\perp$ -free Böhm trees  $\lambda$ -terms by observation depths, and establish the bijective correspondence with infinitary canonical terms of type `ctm`. The validity condition is established similarly by a condition on traces: for each trace in a productive Böhm tree, there is a corresponding trace in the infinitary canonical term of type `ctm`. The productivity corresponds to the validity: each occurrence of a constant implies the occurrences of the `progress` coinductive constructor for each argument on its spine.  $\square$

## 8 CASE STUDY: CO-NATURAL NUMBERS AND CO-BINARY NUMBERS

Recall the definition of unary conatural numbers [Chen and Pfenning 2023], which are exactly natural numbers together with  $\infty$  where  $\infty = \text{cosucc } \infty$ .

```
conat : cotype.
cozero : conat.
cosucc : conat -> conat.
```

Similar to the way we define binary numbers for “little-endian” presentations [Pfenning 2019], where we observe the least significant digits first in an inductive datatype, we define the analogous “little-endian” presentation for conats using binary streams. In this representation, every conatural number that is finite has a unique representation: `b1` and `b0`’s followed by an infinite stack of `b0`’s. The conatural number  $\infty$  has infinitely many representations, as long as `b1`’s occur infinitely often in the stream.<sup>4</sup>

```
cobin : cotype.
b0 : cobin -> cobin.
b1 : cobin -> cobin.
```

**THEOREM 8.1 (ADEQUACY OF ENCODING).** *There exists a bijection between the cobinary numbers and the canonical terms of type `cobin`.*

**PROOF.** We again characterize cobinary numbers by observation, and prove the correspondence directly by induction on the observation depth and the structure of the terms or the structure of cobinary numbers.  $\square$

We have the following examples of cobinary numbers. `bzero` is an encoding of the cobinary 0. `bone` is an encoding of the cobinary number 1. `w1` and `w2` are two examples of the cobinary number  $\infty$ . `bsucc` is an encoding of the “successor” relation between two cobinary numbers.

<sup>4</sup>The reader may notice that the cobinary numbers are defined exactly the same as the real numbers in Section 2.3. Thus, the definition of the `bplus` relation later is also a definition of the sum of two real numbers.

```

bzero : cobin = b0 (bzero).
bone  : cobin = b1 (bzero).
w1    : cobin = b1 (w1).
w2    : cobin = b1 (b0 w2).

```

```

bsucc : cobin -> cobin -> cotype.
bsucc/0 : bsucc (b0 X) (b1 X).
bsucc/1 : bsucc X Y -> bsucc (b1 X) (b0 Y).

```

**THEOREM 8.2 (ADEQUACY OF ENCODING).** *bsucc B1 B2 is inhabited if and only if B2 is a successor of B1.*

**PROOF.** Directly by induction on the observation depth, and the structure of the term or the structure of the informal proof that B2 is a successor of B1.  $\square$

We may define conversions tobin and frombin establishes between unary numbers and binary numbers.

```

coplus : conat -> conat -> conat -> cotype.
coplus/0 : coplus cozero Y Y.
coplus/1 : coplus X Y Z
           -> coplus (cosuss X) Y (cosucc Z).

```

```

tobin : conat -> cobin -> cotype.
tobin/0 : tobin cozero bzero.
tobin/1 :
  tobin X Y
  -> bsucc Y Z
  -> tobin (cosucc X) Z.

```

```

frombin : cobin -> conat -> cotype.
frombin/0 : frombin X Y
           -> coplus Y Y Z
           -> frombin (b0 X) Z.
frombin/1 : frombin X Y
           -> coplus Y Y Z
           -> frombin (b1 X) (cosucc Z).

```

We then define the cobinary plus relation. The cobinary plus sums up two cobinary numbers. The definition will not require a base case as cobinary numbers are defined coinductively.

```

bplus : cobin -> cobin -> cobin -> cotype.
bplus/00 :
  bplus X Y Z ->
  bplus (b0 X) (b0 Y) (b0 (Z)).
bplus/01 :
  bplus X Y Z ->
  bplus (b0 X) (b1 Y) (b1 (Z)).
bplus/10 :
  bplus X Y Z ->
  bplus (b1 X) (b0 Y) (b1 (Z)).
bplus/11 :

```

```

bplus X Y Z ->
bsucc Z W ->
bplus (b1 X) (b1 Y) (b0 W).

```

We show some simple invariants of binary streams, such as that  $0+0 = 0$ , and that the successor of 0 is 1.

```

b0+0is0 : bplus bzero bzero bzero = bplus/00 (b0+0is0).
bsucc0is1 : bsucc bzero bone = bsucc/0.

```

We may show that `bplus` is indeed sound by presenting an encoding of the proof for the following *cobinary sum soundness* theorem: For all cobinary numbers  $N$  and  $M$ , if  $M$  is the “successor” of  $N$ , then the cosuccessor of the unary conatural number corresponding to  $N$  is equal to the cosuccessor of the unary conatural number corresponding to  $M$ . The statement of the theorem is encoded as the `bsucc_sound` type family and the proof is encoded as terms inhabiting the type family. This technique follows the Twelf [Pfenning and Schürmann 1999; Schürmann and Pfenning 2003].

```

eqconat : conat -> conat -> cotype.
eqconat/0 : eqconat cozero cozero.
eqconat/1 : eqconat X Y -> eqconat (cosucc X) (cosucc Y).

eqconat/refl : eqconat X X -> cotype.
eqconat/refl/0 : eqconat/refl eqconat/0.
eqconat/refl/1 : eqconat/refl EQR
  -> eqconat/refl (eqconat/1 EQR).

```

```

bsucc_sound : {N}{M} {CN} {CM} frombin N CN -> frombin M CM
  -> bsucc N M -> eqconat (cosucc CN) CM -> cotype.

```

```

bsucc_sound/0 :
  eqconat/refl EQC
  -> bsucc_sound (b0 N') (b1 N')
    (frombin/0 N'FB N'CP)
    (frombin/1 N'FB N'CP)
    bsucc/0
    (eqconat/1 EQC).

```

```

bsucc_sound/1 :
  bsucc_sound N' M' N'FB M'FB SN'isM' EQC
  ->
  bsucc_sound (b1 N') (b0 M')
    (frombin/1 N'FB N'CP)
    (frombin/0 M'FB M'CP)
    (bsucc/1 SN'isM')
    (eqconat/1 EQC).

```

**THEOREM 8.3 (ADEQUACY OF ENCODING).** *There is a bijection between the informal derivation showing the cobinary sum soundness theorem and the canonical terms of the fully instantiated type family type `bsucc_sound`.*

*That is, there is a bijection between the canonical terms of the type*

$$bsucc\_sound \ulcorner N \urcorner \ulcorner M \urcorner \ulcorner CN \urcorner \ulcorner CM \urcorner \ulcorner SNM \urcorner \ulcorner EQSCNCM \urcorner$$

*and the derivation (proof) of the following theorem: for all cobinary numbers  $N$  and  $M$ , if  $M$  is the “successor” of  $N$  (evidenced by a derivation  $SNM$ ), then  $CN$ , the cosuccessor of the unary conatural*

number corresponding to  $N$ , is equal to  $CM$ , the cosuccessor of the unary conatural number corresponding to  $M$ , evidenced by  $EQSCNCM$ .

PROOF. The derivation for the proof of the theorem is infinite. Then the adequacy proof follows directly by induction on the observation depth, and then the structure of term or the proof.  $\square$

## 9 RELATED WORK

**Observation Depth.** The idea of observation depth is inspired by prior works in handling infinitary structures, such as non-terminating computations in the semantics of programming languages, including step-indexed logical relations [Ahmed 2004], the semantics of mixed induction and coinduction in subtyping systems for recursive types [Lakhani et al. 2022] and in sized types [Abel and Pientka 2016; Somayyajula and Pfenning 2022], the tree topology of Böhm trees [Barendregt 1985] and infinitary lambda calculus [Kennaway et al. 1997; Severi and de Vries 2017], and prior works in the semantics and metatheory of CoLF [Chen 2021; Chen and Pfenning 2023].

**Logical Frameworks.** The use of typed  $\lambda$ -terms to model syntax trees dates back to the logical framework LF [Harper et al. 1993]. The idea of using only  $\beta$ -normal- $\eta$ -long of typed  $\lambda$ -terms is first seen in the development of Concurrent LF [Watkins et al. 2002], along with the technique of hereditary substitution. The metatheory of LF with only canonical terms is further developed in the research of encoding the simply typed  $\lambda$ -calculus and its metatheorems in Twelf [Harper and Licata 2007; Pfenning and Schürmann 1999; Watkins et al. 2002]. Twelf also incorporates a logic programming engine on LF signatures and mechanical checking for metatheorems [Pfenning 1991; Schürmann and Pfenning 2003]. The development of a logic programming interpretation of  $\text{CoLF}^\omega$  will be future work.

**Infinitary  $\lambda$ -Calculus.** Böhm trees were first devised as a method for studying the solvability of  $\lambda$ -terms [Barendregt 1985], and serve to semantically distinguish between  $\lambda$ -terms that do not have a normal form. The  $\perp$ -free Böhm trees are also called hereditary head normal terms [Tatsuta 2008]. Various other forms of infinitary  $\lambda$ -calculus were studied as representations of programs involving recursion [Huet 1998; Kennaway et al. 1997; Severi and de Vries 2017]. However, most studies on infinitary  $\lambda$ -calculus are done in an untyped setting. The notion of productivity comes from Coquand’s work on defining functions on infinitary objects in a type theory [Coquand 1993], and from cut-elimination in linear logic with fixed points [Derakhshan and Pfenning 2019, 2021; Fortier and Santocanale 2013]. The notion of validity follows from a similar definition of CoLF [Chen and Pfenning 2023] and the Horn  $\mu$ -calculus [Charatonik et al. 1998].

**Levels of Priorities.** In  $\text{CoLF}^\omega$ , simple proofs by induction and proofs by coinduction have trivial embedding in this framework. Almost all practical mixed inductive and coinductive structures and mixed inductive and coinductive proofs including subtyping systems [Lakhani et al. 2022], use a simple two-level of priorities, where the induction is nested inside the coinduction. Our definition of infinitary syntax trees follows a similar approach where the observation depth serves as the coinductive progress point. The validity condition, however, assumes an infinite number of levels of priorities, following the prior work of CoLF [Chen and Pfenning 2023] and proof systems with mixed induction and coinduction [Charatonik et al. 1998; Fortier and Santocanale 2013].

## 10 CONCLUSION

We have presented an interpretation of CoLF-style signatures that the term model may be arbitrary finitary and infinitary terms. We take the notion of finitary observation as central to the definition of infinitary structures and characterize the infinite syntax trees inductively in terms of observation depth. The equality on infinitary terms is just a bisimulation on the infinitary terms. We then characterize productivity as a subset of the infinitary syntax trees that are closed under

hereditary substitution. Typing structures are subsequently imposed on productive Böhm trees to define a dependently typed logical framework  $\text{CoLF}^\omega$ . Finally, we have applied  $\text{CoLF}^\omega$  in the encoding of productive Böhm trees and cobinary numbers as case studies.

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