A Saturation-Based Unification Algorithm for Higher-Order Rational Patterns

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Higher-order unification has been shown to be undecidable [Huet 1973]. Miller discovered the pattern fragment and subsequently showed that higher-order pattern unification is decidable and has most general unifiers [1991]. We extend the algorithm to higher-order rational terms (a.k.a. regular Böhm trees [Huet 1998], a form of cyclic λ-terms) and show that pattern unification on higher-order rational terms is decidable and has most general unifiers. We prove the soundness and completeness of the algorithm.

1 INTRODUCTION

Unification is the backbone of logic programming [Miller 1991] and is also used in type reconstruction in the implementation and coverage checking of dependent type theories [Pfenning and Schürmann 1999; Schürmann and Pfenning 2003]. Given a list of equations with unification variables (metavariables), unification is the problem of finding substitutions for the unification variables such that the equations hold true. Often, one is interested in finding most general unifiers. Consider the following equation as an example, where unification variables are written in capital letters.

$$\lambda x.\, \lambda y.\, \lambda z.\, F\, x\, y \doteq \lambda x.\, \lambda y.\, \lambda z.\, G\, y\, z$$

The most general unifier in this case is $F = \lambda v.\, \lambda w.\, H\, v$ and $G = \lambda u.\, \lambda v.\, H\, v$ for some fresh unification variable H.

In this paper, we provide a unification algorithm on higher-order rational terms in the sense of the type theory CoLF [Chen and Pfenning 2023], where two λ-terms are equal if their infinite unfoldings as rational trees are equal. The higher-order rational terms we are considering are also called $\bot$-free regular Böhm trees by Huet [1998]. Our work is distinguished from recent works on nominal unification in λ-calculus with recursive let [Schmidt-Schauß et al. 2022] (a.k.a. cyclic λ-calculus), in that the notion of equality in their work is much weaker than ours. Their equality is based on alpha-equivalence and permutation of order of declaration within the recursive let construct, but our equality is based on the infinite tree equalities generated from circular terms. For instance, given two recursive definitions $r =_d c\, r$ and $s =_d c\, (c\, s)$, our algorithm considers r and s to be equal, whereas Schmidt-Schauß's algorithm distinguishes these two terms.

We only consider the case of unification problems between simply-typed higher-order rational terms, and in particular, we treat validity as a separate issue and thus do not distinguish between type and cotype [Chen and Pfenning 2023]. For instance, when encountering the unification problem $F = \text{succ}\ F$, supposedly with the type of natural numbers, our algorithm is happy to come up with the solution F being an infinite stack of succ's, and disregards the fact the circular terms $F = r, r =_d \text{succ}\ r$ are not valid. In an implementation, validity checking can be a separate procedure from unification.

One of the simplest examples of higher-order unification on higher-order rational terms is the following one, where F is of simple type $(* \rightarrow *) \rightarrow *$.

$$\lambda x.\, x\, (F\, x) \doteq \lambda x.\, F\, x$$

If we were to consider this problem in the setting of non-cyclic unification, there would be no unifier due to the failure of the occurs check. However, our cyclic-unification algorithm will successfully find the unifier $F = \lambda x.\, r\, x$ where $r =_d \lambda x.\, x\, (r\, x)$. The symbol r is a recursion constant that unfolds to the infinitary term, $\lambda x.\, x\, (x\, (x\, (x\, \ldots)))$, an abstraction that binds x and its body is an infinite stack of x's.

We first present the algorithm for first-order rational unification in a new way and then extend the algorithm to include higher-order patterns. Examples use the syntax of Twelf [Pfenning and Schürmann 1999] extended with cyclic terms of CoLF [Chen and Pfenning 2023]. The rest of this paper is structured as follows.

- In Section 2.1, we formally define the problem of unification on first-order rational terms and extend it to higher-order rational terms in Section 3.1.
- In Section 2.2, we describe a preprocessing phase and give an algorithm that flattens recursive definitions for first-order terms, and in Section 3.2 for higher-order rational terms.
- In Section 2.3, we formally define substitutions and unifiers for first-order terms, and in Section 3.3 for higher-order rational terms.
- In Section 2.4, we describe a saturation-based algorithm for first-order terms and extend it to pattern unification on higher-order rational terms in Section 3.4.
- In Section 2.5, we prove the correctness of our algorithm for first-order terms, and in Section 3.7 for higher-order rational terms.

2 FIRST-ORDER RATIONAL UNIFICATION

First-order rational unification [Jaffar 1984] arises directly out of first-order unification [Robinson 1965], but without occurs check. We give a new presentation of the algorithm based on saturation [Pfenning 2006], that mimics the structure of the higher-order case in Section 3.

2.1 Problem Formulation

We present a "concrete syntax" for a unification problem in this section, and then present a "normal form" of the unification problem (in Section 2.2) that the algorithm and the proofs assume. With three syntactic entities, *constants* (written c, d, or e) and *unification variables* (written in capital letters E, F, G, H), *recursion constants* (written r, s, t) [Chen and Pfenning 2023], possibly with subscripts, a first-order *concrete unification context* Δ_c is a system of equations $T \doteq T'$ together with definitions for recursion constants that may occur in T. The grammar is shown as follows. It enforces that recursive definitions are required to be contractive: $r =_d T$ means the head of T must be a constant.

Concrete Unification Contexts	$\Delta_c ::= [] \mid \Delta_c, T_1 \doteq T_2 \mid \Delta_c, r =_d c\, T_1 \, \ldots \, T_n$
Terms	$T ::= c\, T_1 \, \ldots \, T_n \mid H \mid r$

We now define the infinitary denotation of T in a context Δ_c by depth k observations of M. First, define $M_\perp$ to be first-order terms with the symbol $\perp$, and contractive and recursive unification variables (defined later in Section 2.2).

$$M_\perp ::= c\,(M_\perp)_1 \, \ldots \, (M_\perp)_n \mid H^{\blacksquare} \mid H^{\bigcirc} \mid \perp$$

We define *definitional expansion* up to depth k of a term T into $M_\perp$ as the function $\mathsf{exp}^{\Delta_c}_{(k)}(T) = M_\perp$, defined by lexicographic induction on (k, T). Since the parameter Δ_c remains unchanged throughout, we omit writing it to reduce visual clutter if it is not referenced.

$$\begin{aligned}
&\mathsf{exp}_{(0)}(T) = \perp \\
&\mathsf{exp}_{(k+1)}(c\, T_1 \, \ldots \, T_n) = c\,(\mathsf{exp}_{(k)}(M_1)) \, \ldots \, (\mathsf{exp}_{(k)}(M_n)) \\
&\mathsf{exp}_{(k+1)}(H) = H^{\bigcirc} \\
&\mathsf{exp}^{\Delta_c}_{(k+1)}(r) = \mathsf{exp}_{(k+1)}(c\, T_1 \, \ldots \, T_n) \text{ if } r =_d c\, T_1 \, \ldots \, T_n \in \Delta_c
\end{aligned}$$

As an example, given a signature for conatural numbers and their simple equality, we are asked to find which number's double cosuccessor is `omega`.

```
conat : cotype.
cozero : conat.
```

```
cosucc : conat -> conat.

omega : conat = cosucc omega.
?- omega = (cosucc (cosucc H)).
```

We may formulate the problem as follows, where H is a fresh unification variable standing for the answer to our query.

$$\Delta_c = \{omega =_d \ cosucc \ \ omega, cosucc\,(cosucc \ \ H) \doteq omega\}$$

We will not define unifiers for the concrete unification context, but the definition would look similar to the one for the unification context after the preprocessing phase defined next. Eventually, we will find the following unifier for Δ_c.

$$\Gamma_c = \{H \ \doteq \ \ omega, omega =_d \ cosucc \ \ omega\}$$

2.2 Preprocessing

The purpose of preprocessing is to put recursive definitions into shallow forms that are one level deep. This greatly simplifies the termination proof of the unification algorithm here and for the higher-order case, which we eventually wish to develop. Terms are now divided into recursive terms and contractive terms. Similarly, unification variables are divided into *recursive unification variables* (with superscript $^{\bigcirc}$), which may unify with only recursion constants, and *contractive unification variables* (with superscript $^{\blacksquare}$) which may only unify contractive terms. We use the lower case letter m to denote either $\blacksquare$ or $\bigcirc$ and write H^m to indicate a unification variable H that is either contractive or recursive. We also include a special constant contra for contradictory unification contexts that do not have a unifier. The unification context now only permits equations between two recursive terms or two contractive terms. The grammar is as follows.

Unification Contexts	$\Delta, \Gamma ::= [] \mid \Delta, U_1 \doteq U_2 \mid \Delta, N_1 \doteq N_2 \mid \Delta, r =_d U \mid \Delta, \mathrm{contra}$
Contractive Terms	$U ::= c\, N_1 \ \ldots \ N_n \mid H^{\blacksquare}$
Recursive Terms	$N ::= r \mid H^{\bigcirc}$

As an example, we would like the concrete unification context Δ_c defined in the previous section

$$\Delta_c = \{omega =_d \ cosucc \ \ omega, cosucc(cosucc\, H) \doteq omega\}$$

to be processed to the following unification context Δ.

$$\Delta = \{omega =_d \ cosucc \ \ omega, s =_d \ cosucc\, r, r =_d \ cosucc\, H^{\bigcirc}, s \doteq \ omega\}$$

We define a preprocessing translation from concrete unification contexts to unification contexts. We write $\Delta_c \rhd \Delta$ translation of Δ_c to Δ, $T \rhd^{\blacksquare} U \diamond \Delta$ for translating a term T into a contractive term U with a new context Δ, and $T \rhd^{\bigcirc} N \diamond \Delta$ for translating a term T into a recursive term N with a new context Δ. We treat a unification context as an unordered list and may write Δ_1, Δ_2 to join two contexts Δ_1 and Δ_2 with disjoint sets of recursion constants. If the set of recursion constants of Δ_1 is not disjoint from the set of recursion constants of Δ_2, we may consistently rename recursion constants in Δ_2 such that Δ_1, Δ_2 is always defined.

$\boxed{\Delta_c \rhd \Delta}$

$$\frac{}{[] \rhd []}(1) \qquad \frac{\Delta_c \rhd \Delta_1 \qquad T_1 \rhd^{\bigcirc} N_1 \diamond \Delta_2 \qquad T_2 \rhd^{\bigcirc} N_2 \diamond \Delta_3}{\Delta_c, T_1 \doteq T_2 \rhd \Delta_1, \Delta_2, \Delta_3, N_1 \doteq N_2}(2)$$

$$\frac{\Delta_c \rhd \Delta_1 \qquad c\, T_1 \ \ldots \ T_n \rhd^{\blacksquare} U \diamond \Delta_2}{\Delta_c, r =_d c\, T_1 \ \ldots \ T_n \rhd \Delta_1, \Delta_2, r =_d U}(3)$$

$\boxed{T \rhd^{\bigcirc} N \diamond \Delta}$

$$\frac{c\, T_1 \, \dots \, T_n \rhd^{\blacksquare} U \diamond \Delta}{c\, T_1 \, \dots \, T_n \rhd^{\bigcirc} r \diamond (\Delta, r =_d U)}(r \text{ fresh})(4) \qquad \frac{}{H \rhd H^{\bigcirc} \diamond []}(5) \qquad \frac{}{r \rhd r \diamond []}(6)$$

$\boxed{T \rhd^{\blacksquare} U \diamond \Delta}$

$$\frac{\forall_{i,1\le i\le n}. T_i \rhd^{\bigcirc} N_i \diamond \Delta_i}{c\, T_1 \, \dots \, T_n \rhd^{\blacksquare} c\, N_1 \, \dots \, N_n \diamond (\Delta_1, \dots, \Delta_n)}(7) \qquad \text{(No rules for } T = H \text{ or } T = r\text{)}$$

We define the definitional expansion at depth k for a recursive or a contractive term mutually recursively, $\mathsf{exp}^{\Delta}_{(k)}(U) = M_\perp$ and $\mathsf{exp}^{\Delta}_{(k)}(N) = M_\perp$. We take the liberty to omit writing Δ if it remains unchanged throughout and is not referenced.

$$\begin{aligned}
&\mathsf{exp}_{(0)}(U) = \perp \\
&\mathsf{exp}_{(k+1)}(H^{\blacksquare}) = H^{\blacksquare} \\
&\mathsf{exp}_{(k+1)}(c\, N_1 \, \dots \, N_n) = c\, (\mathsf{exp}_{(k)}(N_1)) \, \dots \, (\mathsf{exp}_{(k)}(N_n)) \\
&\mathsf{exp}_{(0)}(N) = \perp \\
&\mathsf{exp}_{(k+1)}(H^{\bigcirc}) = H^{\bigcirc} \\
&\mathsf{exp}^{\Delta}_{(k+1)}(r) = \mathsf{exp}^{\Delta}_{(k+1)}(U) \text{ if } r =_d U \in \Delta
\end{aligned}$$

The translation preserves the definitional expansion of arbitrary depth.

Theorem 2.1. *We have*

(1) If $T \rhd^{\blacksquare} U \diamond \Delta_2$ and $\mathsf{exp}^{\Delta_c}_{(k)}(s) = \mathsf{exp}^{\Delta_1}_{(k)}(s)$ for all s occurring in T, then $\mathsf{exp}^{\Delta_c}_{(k)}(T) = \mathsf{exp}^{\Delta_1,\Delta_2}_{(k)}(U)$.

(2) If $T \rhd^{\bigcirc} N \diamond \Delta_2$ and $\mathsf{exp}^{\Delta_c}_{(k)}(s) = \mathsf{exp}^{\Delta_1}_{(k)}(s)$ for all s occurring in T, then $\mathsf{exp}^{\Delta_c}_{(k)}(T) = \mathsf{exp}^{\Delta_1,\Delta_2}_{(k)}(N)$.

Proof. Simultaneous induction, where (2) may appeal to (1) without a decrease in size. □

Corollary 2.2. *If $\Delta_c \rhd \Delta$, then every equation in Δ_c corresponds to an equation in Δ with equal definitional denotation, and every recursive definition in Δ_c corresponds to a recursive definition in Δ.*

Proof. Directly by structural induction over $\Delta_c \rhd \Delta$. □

It is worth noting that we have assumed all concrete unification variables H are recursive, in the sense that they may unify with a recursion constant. In practice, implementations may want to use the preprocessed forms directly. The concrete form and the translation procedure merely serve as a mechanism to parse the user's input and as a formal explanation of the flattened definitions.

We take the flattened unification context as the "canonical representation" for a unification problem from now on, and we may use the syntax category M for either U or N. We use $\mathsf{defs}(\Delta)$ and $\mathsf{eqs}(\Delta)$ to denote the list of recursive definitions and equations of Δ respectively. Definitional expansion exp does not depend on unification equations but only on recursive definitions, and thus, we have $\mathsf{exp}^{\Delta}_{(k)}(M) = \mathsf{exp}^{\mathsf{defs}(\Delta)}_{(k)}(M)$, for all Δ, k, and M.

2.3 Term Equality and Unifiers

Two terms are equal in a unification context if they have the same definitional expansion, i.e., given $M \doteq M'$ in Δ, we say that M is equal to M' (and thus the equation $M \doteq M'$ *holds*) if $\mathsf{exp}^{\Delta}_{(k)}(M) = \mathsf{exp}^{\Delta}_{(k)}(M')$ for all k. We say that a unification context is *contradiction-free* if contra is not present in the context.

A (simultaneous) substitution is usually understood as a mapping from unification variables to terms. In the case of circular terms, the substitutions may carry recursive definitions. We choose to define substitutions as unification contexts of special forms, where the left-hand sides of all unification equations are unification variables, and the corresponding right-hand sides are their values. We write Γ for substitutions and Δ for ordinary unification contexts. A *substitution* is a contradiction-free unification context where the left-hand side of each unification equation is a unique unification variable. The set of unification variables that occur on the left-hand sides of a substitution Γ is called the *domain* of the substitution and is written $\mathsf{dom}(\Gamma)$. If a substitution contains an equation $H^m \doteq M$, we say that M is the *value* of H^m in Γ. Two substitutions are equal if they have the same domain, and the definitional expansions of the values of each unification variable in their domain are equal, i.e. $\Gamma = \Gamma'$ if $\mathsf{dom}(\Gamma) = \mathsf{dom}(\Gamma')$, and for all $H^m \in \mathsf{dom}(\Gamma)$, $\exp^{\Gamma}_{(k)}(H^m[\Gamma]) = \exp^{\Gamma'}_{(k)}(H^m[\Gamma])$, where $H^m[\Gamma]$ is the value of H^m in Γ, obtained by the substitution operation that will be defined.

As an example, Γ and Γ' below are substitutions with the domain $\{H^{\bigcirc}\}$.

$\Gamma \quad = \{H^{\bigcirc} \doteq \mathit{omega}, \quad \mathit{omega} =_d \mathit{cosucc}\ \mathit{omega}\}$

$\Gamma' \quad = \{H^{\bigcirc} \doteq s, \qquad \mathit{omega} =_d \mathit{cosucc}\ \mathit{omega}, s =_d \mathit{cosucc}\ \mathit{omega}\}$

Moreover, $\Gamma = \Gamma'$ because the expansions of every unification variable in the domain are equal: $H^{\bigcirc}$ expands to $\mathit{cosucc}\,(\mathit{cosucc} \ldots)$.

We emphasize that in a substitution, unification variables occurring on the right-hand sides of unification equations and in recursive definitions are free. Thus, the substitution Γ'' below has $H^{\bigcirc}$ in the recursive definition free, and Γ'' is not equal to Γ defined above.

$\Gamma'' = \{H^{\bigcirc} \doteq \mathit{omega}, \mathit{omega} =_d \mathit{cosucc}\ H^{\bigcirc}\}$

We write $U[\Gamma]$ and $N[\Gamma]$ for applying the substitution to terms. They are defined in obvious ways.

$$(c\, N_1\ \ldots\ N_n)[\Gamma] = c\,(N_1[\Gamma])\ \ldots\ (N_n[\Gamma])$$

$$(H^{\blacksquare})[\Gamma] = \begin{cases} U' & \text{if } H^{\blacksquare} \doteq U' \in \mathsf{eqs}(\Gamma) \\ H^{\blacksquare} & \text{otherwise} \end{cases}$$

$$(r)[\Gamma] = r$$

$$(H^{\bigcirc})[\Gamma] = \begin{cases} N' & \text{if } H^{\bigcirc} \doteq N' \in \mathsf{eqs}(\Gamma) \\ H^{\bigcirc} & \text{otherwise} \end{cases}$$

The application of substitution Γ to a *unification context* Δ is denoted $\Delta[\Gamma]$, which replace occurrences of unification variables in Γ by their values in Δ, while combining all recursive definitions and performing recursion constant renaming as necessary.

$$\Delta[\Gamma] = \mathsf{defs}(\Gamma), \{M[\Gamma] \doteq M'[\Gamma] \mid M \doteq M' \in \mathsf{eqs}(\Delta)\}, \{r =_d U[\Gamma] \mid r =_d U \in \mathsf{defs}(\Delta)\}$$

The application of a substitution Γ_2 to another *substitution* is Γ_1 is denoted $\Gamma_1[\Gamma_2]$, and it replaces the occurrences of unification variables in the right-hand sides and recursive definitions of Γ_1 by their values in Γ_2 and combine all recursive definitions, performing constant renaming as necessary.

$$\Gamma_1[\Gamma_2] = \mathsf{defs}(\Gamma_2), \{H^m \doteq M'[\Gamma_2] \mid H^m \doteq M' \in \mathsf{eqs}(\Gamma_1)\}, \{r =_d U[\Gamma_2] \mid r =_d U \in \mathsf{defs}(\Gamma_1)\}$$

The composition of substitutions is denoted $\Gamma_1 \circ \Gamma_2$ (applying Γ_1 and then applying Γ_2), is defined to be $\Gamma_1[\Gamma_2]$ plus any additional substitutions in Γ_2.

$$\Gamma_1 \circ \Gamma_2 = (\Gamma_1[\Gamma_2]), \{H^m \doteq M \mid H^m \doteq M \in \mathsf{eqs}(\Gamma_2) \wedge H^m \notin \mathsf{dom}(\Gamma_1)\}$$

Let $UV(\Delta)$ denote the set of all unification variables that occur in Δ, a *unifier* for a contradiction-free unification context Δ is a substitution Γ such that $UV(\Delta) = \mathsf{dom}(\Gamma)$, and every equation in $\Delta[\Gamma]$ holds. A unification context Δ with contra $\in \Delta$ has no unifiers. A unifier Γ_1 is *more general* than another unifier Γ_2 if there is a substitution Γ' such that $\Gamma_1 \circ \Gamma' = \Gamma_2$.

As an example, given Γ and Δ defined below, Γ is a unifier of Δ, because every equation in $\Delta[\Gamma]$ holds. Notice that when carrying out the substitution, the duplicate recursion constant *omega* in Γ is renamed to t. The major changes are highlighted in blue.

$\Gamma = \{$	$\Delta = \{$	$\Delta[\Gamma] = \{$
$H^{\bigcirc} \doteq omega,$	$omega =_d cosucc\ omega,$	$omega =_d cosucc\ omega,$
	$s =_d cosucc\ r,$	$s =_d cosucc\ r,$
	$r =_d cosucc\ H^{\bigcirc},$	$r =_d cosucc\ t,$
$omega =_d cosucc\ omega$		$t =_d cosucc\ t,$
$\}$	$s \doteq omega$	$s \doteq omega$
	$\}$	$\}$

2.4 The Unification Algorithm

We saturate the unification context Δ using the rules defined below. If all the premises of a rule are present in the context, we add the rule's conclusion to the context. The algorithm terminates when no new equations or recursive definitions can be added to the context. The goal of the rules is to ensure that in a saturated unification context, either contra is present, indicating there is no unifier, or there is an equation between each unification variable and its value in a unifier.

Structural Rules:

$$\frac{c\, N_1\ \ldots\ N_n \doteq d\, N_1'\ \ldots\ N_n'}{\text{contra}}\,(c \neq d)(1) \qquad \frac{c\, N_1\ \ldots N_n \doteq c\, N_1'\ \ldots N_n'}{N_1 \doteq N_1', \ldots, N_n \doteq N_n'}\,(2)$$

Expansion, Symmetry, and Transitivity

$$\frac{r \doteq s \qquad r =_d U_1 \qquad s =_d U_2}{U_1 \doteq U_2}\,(3) \qquad \frac{U \doteq U'}{U' \doteq U}\,(4) \qquad \frac{U_1 \doteq U_2 \qquad U_2 \doteq U_3}{U_1 \doteq U_3}\,(5) \qquad \frac{N \doteq N'}{N' \doteq N}\,(6)$$

$$\frac{N_1 \doteq N_2 \qquad N_2 \doteq N_3}{N_1 \doteq N_3}\,(7)$$

We give an example of the ways the algorithm operates on our previous example. We label each equation with a number and use Δ_i to refer to the set of equations $(1) - (i)$. For example, our example Δ is denoted Δ_4, consisting of equations $(1) - (4)$. At each step, we show some additional equations and the ways they are obtained. We only show the first few important steps and the rest will be only symmetry and transitivity.

(1) $omega =_d cosucc\ omega$	given
(2) $s =_d cosucc\ r$	
(3) $r =_d cosucc\ H^{\bigcirc}$	
(4) $s \doteq omega$	
(5) $cosucc\ r \doteq cosucc\ omega$	by Rule (3) on (4), (2) and (1)
(6) $r \doteq omega$	by Rule (2) on (5)
(7) $cosucc\ H^{\bigcirc} \doteq cosucc\ omega$	by Rule (3) on (6), (3) and (1)
(8) $H^{\bigcirc} \doteq omega$	by Rule (2) on (7)
(9) ...	by Rules (4)(5)(6)(7) on all equations

We now describe how a unifier may be constructed from a saturated contradiction-free unification context. Given a unification context Δ, we say that a unification variable $H^{\blacksquare}$ is *resolved* if there is an equation of the form $H^{\blacksquare} \doteq c\, N_1\ \ldots\ N_n$ or $c\, N_1\ \ldots\ N_n \doteq H^{\blacksquare}$, and $c\, N_1\ \ldots\ N_n$ is called a resolution of $H^{\blacksquare}$. Similarly, we say that a unification variable $H^{\bigcirc}$ is *resolved* if there is an equation of the form $H^{\bigcirc} \doteq r$ or $r \doteq H^{\bigcirc}$, and r is called a resolution of $H^{\bigcirc}$. In a unification context, every unification variable is either resolved or unresolved. There may be multiple resolutions for each resolved unification variable, we pick a unique resolution for each unification variable. The choice of resolution is not important, because every resolution will be equal modulo definitional expansion in a saturated contradiction-free context. Unresolved unification variables form an equivalence class equated by $\doteq$, and we pick a unique representative variable for each class. We construct the substitution $\Gamma = \mathsf{unif}(\Delta)$ for a contradiction-free context Δ as follows.

(1) Start with Γ containing all recursive definitions of Δ.
(2) For each resolved unification variable in $UV(\Delta)$, add to Γ the unification variable and its resolution.
(3) For each unresolved unification variable in $UV(\Delta)$, add to Γ the unification variable and the representative unification variable for its equivalence class.
(4) Replace the occurrences of resolved unification variables in the right-hand sides and recursive definitions of Γ with their resolutions, and replace the occurrences of unresolved unification variables in the right-hand sides and recursive definitions of Γ with their representative unification variables. Repeat this step until all unification variables in the right-hand sides and recursive definitions are representative unification variables for some equivalence class of unresolved unification variables.

We will later show that if Δ is a saturated contradiction-free unification context, then $\Gamma = \mathsf{unif}(\Delta)$ is a unifier for Δ. As an example, we show how the unifier for Δ_8 (equations (1) – (8) defined above) can be constructed. The main differences in each step are highlighted in blue.

(1) Initialize Γ_1 to all recursive definitions of Δ_8.
(2) Since $H^{\bigcirc}$ is resolved, we add its resolution to get Γ_2.
(3) There is no unresolved unification variable, we skip step (3).
(4) Replace occurrences of resolved unification variables with their resolutions to get Γ_3.

$$\begin{array}{l} \Gamma_1 = \{ \\ \quad \mathit{omega} =_d \mathit{cosucc}\ \mathit{omega}, \\ \quad s =_d \mathit{cosucc}\ r, \\ \quad r =_d \mathit{cosucc}\ H^{\bigcirc} \\ \} \end{array} \quad \Bigg| \quad \begin{array}{l} \Gamma_2 = \{ \\ \quad \mathit{omega} =_d \mathit{cosucc}\ \mathit{omega}, \\ \quad s =_d \mathit{cosucc}\ r, \\ \quad r =_d \mathit{cosucc}\ H^{\bigcirc}, \\ \quad H^{\bigcirc} \doteq \mathit{omega} \\ \} \end{array} \quad \Bigg| \quad \begin{array}{l} \Gamma_3 = \{ \\ \quad \mathit{omega} =_d \mathit{cosucc}\ \mathit{omega}, \\ \quad s =_d \mathit{cosucc}\ r, \\ \quad r =_d \mathit{cosucc}\ \mathit{omega}, \\ \quad H^{\bigcirc} \doteq \mathit{omega} \\ \} \end{array}$$

(5) Note that we may remove unused recursive definitions (s, r) to get an equivalent substitution Γ_4.
$\Gamma_4 = \{\mathit{omega} =_d \mathit{cosucc}\ \mathit{omega}, H^{\bigcirc} \doteq \mathit{omega}\}$

2.5 Correctness of the Algorithm

We want to show that given a unification context Δ_1, it has a finitary saturation sequence

$$\Delta_1 \longrightarrow \Delta_2 \longrightarrow \cdots \longrightarrow \Delta_n$$

where Δ_n is a saturated unification context that has $\mathsf{unif}(\Delta_n)$ as its most general unifier. Moreover, unifiers are preserved between Δ_i and Δ_{i+1}. Then, the most general unifiers of Δ_n are the most general unifiers of Δ_1.

Theorem 2.3 (Termination). *The saturation algorithm always terminates.*

Proof. We observe that all terms in an equation are built up from recursion constants, unification variables, and constants, and all terms have finite depth (due to the grammar) and finite width (the maximum width is preserved by the algorithm). There can be only finitely many equations given a bounded number of recursion constants, constants, and unification variables, and there are no rules that create additional recursion constants, constants, or unification variables.

□

Theorem 2.4 (Unifiers for Saturated Unification Contexts). *Given any saturated contradiction-free unification context Δ, let $\Gamma = \mathsf{unif}(\Delta)$, then Γ is a unifier for Δ. Moreover, it is the most general unifier.*

Proof. To show Γ is a unifier, we need to show that $\mathsf{dom}(\Gamma) = UV(\Delta)$, which is true by definition, and that every equation in $\Delta[\Gamma]$ holds. We show the following two claims simultaneously by induction on k, where claim (2) may refer to claim (1) without decreasing k.

(1) For all $U_1 \doteq U_2$ in Δ, $\mathsf{exp}^{\Delta[\Gamma]}_{(k)}(U_1[\Gamma]) = \mathsf{exp}^{\Delta[\Gamma]}_{(k)}(U_2[\Gamma])$.
(2) For all $N_1 \doteq N_2$ in Δ, $\mathsf{exp}^{\Delta[\Gamma]}_{(k)}(N_1[\Gamma]) = \mathsf{exp}^{\Delta[\Gamma]}_{(k)}(N_2[\Gamma])$.

Both claims are trivial when $k = 0$. Consider the case when $k > 0$, we show (1) and (2) by case analysis on the structure of $U_1 \doteq U_2$ and $N_1 \doteq N_2$.

(a) Both U_1 and U_2 have constants as their heads. Since $\mathsf{contra} \notin \Delta$, they must have identical constant heads. Now let $U_1 = c\, N_1 \ldots N_n$ and $U_2 = c\, N_1' \ldots N_n'$. Since Δ is saturated, we have $N_i \doteq N_i'$ for all $1 \le i \le n$. The result then follows from the fact that each N_i and N_i' have equal definitional expansion up to depth $k - 1$ by induction hypothesis.
(b) Both U_1 and U_2 are contractive unification variables. Due to saturation, either both are unresolved, and the result follows because they would be in the same equivalence class and thus have the same representative unification variable, or both are resolved. If they have a unique resolution U, then we have $\mathsf{exp}_{(k)}(U_1) = \mathsf{exp}_{(k)}(U) = \mathsf{exp}_{(k)}(U_2)$. If they have multiple resolutions and one of the resolutions is U, saturation guarantees that there is an equation between every resolution. Rule (2) ensures that the equations between children of the head constants are in Δ, and the two terms would be equal by IH, using a similar technique as the case (a).
(c) One of U_1 and U_2 is a unification variable, and the other has a constant as its head. Obviously this is a resolution equation, and it suffices to show that all other resolutions have equal definitional expansions up to depth k, which follows from saturation and the case (a).
(d) Both N_1 and N_2 are recursion constants. Let $N_1 = r$, where $r =_d U_1 \in \Delta$ and $N_2 = s$, where $s =_d U_2 \in \Delta$. Since Δ is saturated, $U_1 \doteq U_2 \in \Delta$, and by IH (i.e. case (a) above), $\mathsf{exp}^{\Delta[\Gamma]}_{(k)}(U_1[\Gamma]) = \mathsf{exp}^{\Delta[\Gamma]}_{(k)}(U_2[\Gamma])$, and $\mathsf{exp}^{\Delta[\Gamma]}_{(k)}(r[\Gamma]) = \mathsf{exp}^{\Delta[\Gamma]}_{(k)}(r) = \mathsf{exp}^{\Delta[\Gamma]}_{(k)}(U_1[\Gamma]) = \mathsf{exp}^{\Delta[\Gamma]}_{(k)}(U_2[\Gamma]) = \mathsf{exp}^{\Delta[\Gamma]}_{(k)}(s) = \mathsf{exp}^{\Delta[\Gamma]}_{(k)}(s[\Gamma])$.
(e) The case when either or both of N_1 and N_2 are recursive unification variables are exactly analogous to cases (b) and (c).

To show Γ is the most general unifier, given any other unifier Γ_2 of Δ, it suffices to construct a unifier Γ_1 such that $\Gamma \circ \Gamma_1 = \Gamma_2$. But the construction of Γ_1 is easy: Γ_2 must map resolved unification variables analogously as Γ (otherwise a contradiction will arise), and it may choose to map equivalence classes of unresolved unification variables freely. Γ_1 simply records how unresolved unification variables are mapped in Γ_2.

□

Lemma 2.5. *Let Δ' be a unification context, and let Δ have the same set of recursive definitions and unification variables, but fewer unification equations than Δ', i.e.,* $\mathrm{eqs}(\Delta) \subseteq \mathrm{eqs}(\Delta')$, $\mathrm{defs}(\Delta) = \mathrm{defs}(\Delta')$, $UV(\Delta) = UV(\Delta')$, *then any unifier of Δ' is a unifier of Δ.*

Proof. Let Γ be a unifier of Δ', all unification equations of $\Delta'[\Gamma]$ hold. Take any $M \doteq M' \in \Delta$, we know $\exp_{(k)}^{\Delta'[\Gamma]}(M[\Gamma]) = \exp_{(k)}^{\Delta'[\Gamma]}(M'[\Gamma])$, we have $UV(\Delta) = UV(\Delta')$, and it suffices to show $\exp_{(k)}^{\Delta[\Gamma]}(M[\Gamma]) = \exp_{(k)}^{\Delta[\Gamma]}(M'[\Gamma])$ by showing $\exp_{(k)}^{\Delta[\Gamma]}(M[\Gamma]) = \exp_{(k)}^{\Delta'[\Gamma]}(M[\Gamma])$. But since definitional expansions only depend on recursive definitions, we have

$$\exp_{(k)}^{\Delta[\Gamma]}(M[\Gamma]) = \exp_{(k)}^{\mathrm{defs}(\Delta[\Gamma])}(M[\Gamma]) = \exp_{(k)}^{\mathrm{defs}(\Delta'[\Gamma])}(M[\Gamma]) = \exp_{(k)}^{\Delta'[\Gamma]}(M[\Gamma])$$

□

Lemma 2.6. *If Γ is a unifier for Δ, then Γ is a unifier for Δ' where Δ' has all recursive definitions of Δ and additional true equations $M \doteq M'$ in the sense that* $\exp_{(k)}^{\Delta[\Gamma]}(M[\Gamma]) = \exp_{(k)}^{\Delta[\Gamma]}(M'[\Gamma])$.

Proof. Because definitional expansion depends only on recursive equations but not unification equations, we have $\exp_{(k)}^{\Delta[\Gamma]}(M) = \exp_{(k)}^{\Delta'[\Gamma]}(M)$ for all k and M. □

Theorem 2.7 (Correspondence). *If Δ' is obtained from Δ by applying one of the rules, then the unifiers of Δ' and the unifiers of Δ coincide.*

Proof. We analyze each rule.

Case (1), both Δ and Δ' have no unifiers.

Case (2), it's easy to check that any unifier Γ of Δ' is a unifier of Δ by Lemma 2.5. Now suppose Γ is a unifier of Δ, we want to show that Γ is a unifier for Δ'. The additional equations $M_i \doteq M_i'$ in Δ' satisfy $\exp_{(k)}^{\Delta[\Gamma]}(M_i[\Gamma]) = \exp_{(k)}^{\Delta[\Gamma]}(M_i'[\Gamma])$, and the rest follows by Lemma 2.6.

The rest of the cases are similar to Case (2).

□

3 HIGHER-ORDER PATTERN UNIFICATION

3.1 Problem Formulation

We now give a similar development by allowing recursion constants and unification variables to carry patterns [Miller 1991]: a recursion constant or a unification variable may be applied to a list of pairwise distinct bound variables (written x, y, or z). Here's an example of a higher-order pattern unification problem (without recursive definitions).

$$\lambda x.\, \lambda y.\, \lambda z.\, c\ (F\, x\, y) \doteq \lambda x.\, \lambda y.\, \lambda z.\, c\ (G\, y\, z)$$

A variable may not appear free in a unifier. For instance, the substitution $F\, x\, y = x, G\, y\, z = x$ (i.e. $F = \lambda x.\, \lambda y.\, x, G = \lambda y.\, \lambda z.\, x$) is not a unifier because x is free in the substitution of G but the substitution $F\, x\, y = d, G\, y\, z = d$ is a unifier.

Regular Böhm trees [Huet 1998], subsequently termed higher-order rational terms, provide a natural model for higher-order terms. As with the first-order case, our use of a context containing recursive definitions for recursion constants follows the design of CoLF [Chen and Pfenning 2023]. While CoLF allows repetitions of bound variables in the arguments to recursion constants, we disallow them in the setting of unification to ensure that most general unifiers exist. This is not a restriction in practice, because any recursive definition with repetitive arguments can be rewritten to definitions within the pattern fragment, as observed by Huet [1998]. We assume that every unification variable, recursion constant, constant, or variable is assigned a simple type, and terms are always written in β-normal-η-long forms, except that arguments to recursion constants and unification variables are written in non-expanded forms. We also assume that λ-bound variables may undergo α-renaming. Here's the grammar for the unification problem in concrete syntax.

$$
\begin{array}{lll}
\text{Concrete Unification Contexts} & \Delta_c ::= [] \mid \Delta_c, T_1 \doteq T_2 \mid \Delta_c, r =_d \lambda x_1. \ldots \lambda x_l.\, h\, T_1 \ \ldots \ T_n \\
\text{Terms} & T ::= \lambda x_1. \ldots .\lambda x_l.\, h\, T_1 \ \ldots \ T_n \mid \lambda x_1. \ldots .\lambda x_l.\, H\, y_1 \ \ldots \ y_n \\
 & \quad \mid \lambda x_1. \ldots .\lambda x_l.\, r\, y_1 \ \ldots \ y_n \\
\text{Constant or Variable Heads} & h ::= c \mid x
\end{array}
$$

To avoid visual clutter when writing down a list of terms, we adopt the following conventions of using overlines to represent a list of terms.

(1) A list of variables $\bar{x}$ means $x_1, \ldots, x_l$ that are pairwise distinct.
(2) A list of variables appearing in a binder position means iterative abstractions. For example, $\lambda\bar{x}$ means $\lambda x_1. \ldots .\lambda x_l.$.
(3) A list of variables in an application means iterative applications. For example, $c\,\bar{x}$ means $c\, x_1 \ \ldots \ x_n$. Similarly, a list of terms in an application position means iterative applications. For example, $h\,\overline{N}$ means $h\, N_1 \ \ldots \ N_n$.
(4) The notation $[\bar{y}/\bar{x}]M$ denotes the simultaneous renaming of variables, substituting $\bar{y}$ for $\bar{x}$ in M.

With the new abbreviation notation, the grammar for the concrete syntax for a unification problem may be written as the following.

$$
\begin{array}{ll}
\text{Concrete Unification Contexts} & \Delta_c ::= [] \mid \Delta_c, T_1 \doteq T_2 \mid \Delta_c, r =_d \lambda\bar{x}.\, h\,\overline{T} \\
\text{Terms} & T ::= \lambda\bar{x}.\, h\,\overline{T} \mid \lambda\bar{x}.\, H\,\bar{y} \mid \lambda\bar{x}.\, r\,\bar{y} \\
\text{Constant or Variable Heads} & h ::= c \mid x
\end{array}
$$

The grammar enforces that the definition for a recursion constant is required to be contractive: it has a variable or a constant for its head. We use $FV(T)$ to denote the set of free variables in T. We require all recursive definitions to be closed in the sense that $r =_d \lambda\bar{x}.\, h\,\overline{T} \in \Delta_c$ implies that $FV(h\,\overline{T}) \subseteq \bar{x}$.

As with the first-order case, we define the infinitary denotation of T in a context Δ_c by depth k observations of M. Now $M_\perp$ includes λ-bindings and variables.

$$M_\perp ::= \lambda\bar{x}.\, y\,\overline{M_\perp} \mid \lambda\bar{x}.\, c\,\overline{M_\perp} \mid \lambda\bar{x}.\, H^{\blacksquare}\,\overline{M_\perp} \mid \lambda\bar{x}.\, H^{\bigcirc}\,\overline{M_\perp} \mid \perp$$

We define *definitional expansion* up to depth k of a term T into $M_\perp$ as the function $\exp^{\Delta_c}_{(k)}(T) = M_\perp$, defined by lexicographic induction on (k, T). We omit Δ_c to reduce visual clutter if it is not referenced.

$$
\begin{array}{l}
\exp_{(0)}(T) = \perp \\
\exp_{(k+1)}(\lambda\bar{x}.\, h\, T_1 \ \ldots \ T_n) = \lambda\bar{x}.\, h\,(\exp_{(k)}(T_1)) \ \ldots \ (\exp_{(k)}(T_n)) \\
\exp_{(k+1)}(\lambda\bar{x}.\, H\,\bar{y}) = \lambda\bar{x}.\, H^{\bigcirc}\bar{y} \\
\exp^{\Delta_c}_{(k+1)}(\lambda\bar{x}.\, r\,\bar{y}) = \exp^{\Delta_c}_{(k+1)}(\lambda\bar{x}.\, [\bar{y}/\bar{z}](h\,\overline{T})) \text{ if } r =_d \lambda\bar{z}.\, h\,\overline{T} \in \Delta_c
\end{array}
$$

As an example, we use an encoding of stream processors sp [Ghani et al. 2009], along the lines of Abel and Pientka [2016]; Danielsson and Altenkirch [2010]. At each step, a stream processor may choose to consume an input element or produce an output element and may do so indefinitely. [1]

```
sp : cotype.
element : type.
get: (element -> sp) -> sp.
put: element -> sp -> sp.
```

[1] Stream processors were used to illustrate the semantics of mixed-induction and coinduction, but here we consider stream processors to be purely coinductive. Thus, we are happy to accept stream processors that keep consuming inputs without producing an output.

We may define stream processors odd or even that return only the odd-indexed or even-indexed elements, where the index starts from 0. We write λ-abstractions in square brackets, following the convention of CoLF [Chen and Pfenning 2023].

```
odd : sp = get ([x] even).
even : sp = get ([x] put x odd).
```

We may use unification to determine what is the behavior of the stream processor S after reading two elements of the input, if it behaves the same as odd.

```
?- get ([x] get ([y] S x y)) = odd.
```

The problem may be posed as the following concrete unification context Δ_c, which will be used as a running example.

$$\Delta_c = \{get\,(\lambda x.\ get\,(\lambda y. S\,x\,y)) \doteq odd, odd =_d get\,(\lambda x.\ even), even =_d get\,(\lambda x.\ put\ x\ odd)\}$$

Eventually, we will find the following most general unifier, written in the concrete syntax.

$$\Gamma_c = \{S\,x\,y \doteq r_3\ y, odd =_d get\,(\lambda x.\ get\,(\lambda y. r_3\,y)), r_3 =_d \lambda w.\ put\ w\ odd, \}$$

3.2 Preprocessing

As with the first-order case, we preprocess the unification problem Δ_c such that every recursive definition will only be one level deep. In the higher-order case, this processing is similar to Huet's treatment of regular Böhm trees [1998]. Terms are divided into contractive terms U, which have constants, bound variables, or contractive unification variables as their heads, and recursive terms N, which have either recursion constants or recursive unification variables as their heads. It is still the case that the terms are always written in β-normal-η-long forms, with the exception that arguments to recursion constants and unification variables are written in non-expanded forms. The grammar for terms in their preprocessed form is summarized as follows:

Unification Contexts	$\Delta ::= [] \mid \Delta, U_1 \doteq U_2 \mid \Delta, N_1 \doteq N_2 \mid \Delta, r =_d U \mid \Delta, \text{contra}$
Contractive Terms	$U ::= \lambda\bar{x}.\, y\,\overline{N} \mid \lambda\bar{x}.\, c\,\overline{N} \mid \lambda\bar{x}.\, H^{\blacksquare}\,\bar{y}$
Recursive Terms	$N ::= \lambda\bar{x}.\, r\,\bar{y} \mid \lambda\bar{x}.\, H^{\bigcirc}\,\bar{y}$

We use the letter h to denote either constants c or variables x, but not unification variables. We use $FV(U)$ or $FV(N)$ to denote the set of free variables in U or N, and may use the syntax category M to denote either U or N. We also require all recursive definitions to be closed in the sense that $r =_d U \in \Delta$ implies $FV(U) = \emptyset$. As with the first-order case, $UV(\Delta)$ denotes the set of unification variables in Δ. defs(Δ) and eqs(Δ) denote the list of recursive definitions and equations of Δ respectively. Δ_1, Δ_2 denotes the union of two contexts Δ_1 and Δ_2, consistently renaming recursion constants in Δ_2 if necessary. Δ is *contradiction-free* if contra $\notin \Delta$.

As with first-order terms, we use the judgments $\Delta_c \triangleright \Delta$, $T \triangleright^{\blacksquare} U \diamond \Delta$, $T \triangleright^{\bigcirc} N \diamond \Delta$ for translating from concrete syntax into unification contexts, contractive terms, and recursive terms. They are defined as follows.

$\boxed{\Delta_c \triangleright \Delta}$

$$\frac{}{[] \triangleright []}(1) \qquad \frac{\Delta_c \triangleright \Delta_1 \qquad T_1 \triangleright^{\bigcirc} N_1 \diamond \Delta_2 \qquad T_2 \triangleright^{\bigcirc} N_2 \diamond \Delta_3}{\Delta_c, T_1 \doteq T_2 \triangleright \Delta_1, \Delta_2, \Delta_3, N_1 \doteq N_2}(2)$$

$$\frac{\Delta_c \triangleright \Delta_1 \qquad \lambda\bar{x}.\, h\,\overline{T} \triangleright^{\blacksquare} U \diamond \Delta_2}{\Delta_c, r =_d \lambda\bar{x}.\, h\,\overline{T} \triangleright \Delta_1, \Delta_2, r =_d U}(3)$$

$\boxed{T \rhd^{\bigcirc} N \diamond \Delta}$

$$\frac{h\,\overline{T} \rhd^{\blacksquare} U \diamond \Delta \qquad \bar{z} = FV(h\,\overline{T})}{\lambda\bar{x}.\, h\,\overline{T} \rhd^{\bigcirc} \lambda\bar{x}.\, r\,\bar{z} \diamond (\Delta, r =_d \lambda\bar{z}.\, U)}(r \text{ fresh})(4) \qquad \frac{}{\lambda\bar{x}.\, H\,\bar{y} \rhd \lambda\bar{x}.\, H^{\bigcirc}\,\bar{y} \diamond []}(5)$$

$$\frac{}{\lambda\bar{x}.\, r\,\bar{y} \rhd \lambda\bar{x}.\, r\,\bar{y} \diamond []}(6)$$

$\boxed{T \rhd^{\blacksquare} U \diamond \Delta}$

$$\frac{\forall_{i, 1 \le i \le n}. T_i \rhd^{\bigcirc} N_i \diamond \Delta_i}{\lambda\bar{x}.\, h\,T_1\,\ldots\,T_n \rhd^{\blacksquare} \lambda\bar{x}.\, h\,N_1\,\ldots\,N_n \diamond (\Delta_1, \ldots, \Delta_n)}(7) \qquad \text{(No rules for } T = \lambda\bar{x}.\, H\,\bar{y} \text{ or } T = \lambda\bar{x}.\, r\,\bar{y})$$

As an example, we show how the unification problem Δ_c in the previous section is translated into Δ. Notice that the left-hand-side of the unification equation $get\,(\lambda x.\ get\,(\lambda y.\ S\ x\,y))$ is moved into a recursive definition r_1 according to the definition, and the body of *even* is extracted to r_3.

$\Delta_c = \{$	$\Delta = \{$
$get\,(\lambda x.\ get\,(\lambda y.\, S\,x\,y)) \doteq odd,$	$r_1 \doteq odd,$
	$r_1 =_d get\,(\lambda x.\, r_2\,x),$
	$r_2 =_d \lambda x.\ get\,(\lambda y.\, S^{\bigcirc}\,x\,y),$
$odd =_d get\,(\lambda x.\ even),$	$odd =_d get\,(\lambda x.\ even),$
$even =_d get\,(\lambda x.\ put\ x\ odd)$	$even =_d get\,(\lambda x.\, r_3\,x),$
$\}$	$r_3 =_d \lambda x.\ put\,(r_4\,x)\ odd,$
	$r_4 =_d \lambda x.\, x$
	$\}$

As with the first-order case, we define the definitional expansion at depth k for a recursive or a contractive term mutually recursively, $\exp^{\Delta}_{(k)}(U) = M_\perp$ and $\exp^{\Delta}_{(k)}(N) = M_\perp$. We also take the liberty to omit writing Δ if it remains unchanged throughout and is not referenced.

$$\begin{aligned}
&\exp_{(0)}(U) = \perp\\
&\exp_{(k+1)}(\lambda\bar{x}.\, H^{\blacksquare}\,\bar{y}) = \lambda\bar{x}.\, H^{\blacksquare}\bar{y}\\
&\exp_{(k+1)}(\lambda\bar{x}.\, h\,T_1\,\ldots\,T_n) = \lambda\bar{x}.\, h\,(\exp_{(k)}(T_1))\,\ldots\,(\exp_{(k)}(T_n))\\
&\exp_{(0)}(N) = \perp\\
&\exp_{(k+1)}(\lambda\bar{x}.\, H^{\bigcirc}\,\bar{y}) = \lambda\bar{x}.\, H^{\bigcirc}\bar{y}\\
&\exp^{\Delta_c}_{(k+1)}(\lambda\bar{x}.\, r\,\bar{y}) = \exp^{\Delta_c}_{(k+1)}(\lambda\bar{x}.\, [\bar{y}/\bar{z}](h\,\overline{T})) \text{ if } r =_d \lambda\bar{z}.\, h\,\overline{T} \in \Delta_c
\end{aligned}$$

We show that the translation preserves the definitional expansion of arbitrary depth.

Theorem 3.1. *We have*

(1) If $T \rhd^{\blacksquare} U \diamond \Delta_2$ *and* $\exp^{\Delta_c}_{(k)}(\lambda\bar{x}.\, s\,\bar{y}) = \exp^{\Delta_1}_{(k)}(\lambda\bar{x}.\, s\,\bar{y})$ *for all* s *occurring in* T*, then* $\exp^{\Delta_c}_{(k)}(T) = \exp^{\Delta_1,\Delta_2}_{(k)}(U)$.

(2) If $T \rhd^{\bigcirc} N \diamond \Delta_2$ *and* $\exp^{\Delta_c}_{(k)}(\lambda\bar{x}.\, s\,\bar{y}) = \exp^{\Delta_1}_{(k)}(\lambda\bar{x}.\, s\,\bar{y})$ *for all* s *occurring in* T*, then* $\exp^{\Delta_c}_{(k)}(T) = \exp^{\Delta_1,\Delta_2}_{(k)}(N)$.

Proof. Simultaneous induction on (k, T), where (2) may appeal to (1) without a decrease in size.

We show the case for rule (4) as an example. The premise $h\,\overline{T} \rhd^{\blacksquare} U \diamond \Delta$ implies, by the induction hypothesis, that $\exp^{\Delta_c}_{(k)}(h\,\overline{T}) = \exp^{\Delta_1,\Delta}(U)$. Let $\bar{z} = FV(h\,\overline{T})$, we want to show that

$\exp^{\Delta_c}_{(k)}(\lambda\bar{x}.\, h\,\overline{T}) = \exp^{\Delta_1,\Delta,r=_d\lambda\bar{z}.\,U}_{(k)}(\lambda\bar{x}.\, r\,\bar{z})$. Observe that both $\exp^{\Delta_c}_{(k)}$ and $\exp^{\Delta}_{(k)}$ commute with λ-abstractions, and that $\exp^{\Delta}_{(k)}$ is fixed by the recursive definitions for recursion constants that occur in the argument, i.e. $\exp^{\Delta_1,\Delta}_{(k)}(U) = \exp^{\Delta_1,\Delta,r=_d\lambda\bar{z}.\,U}_{(k)}(U)$. We have $\exp^{\Delta_c}_{(k)}(\lambda\bar{x}.\, h\,\overline{T}) = \lambda\bar{x}.\ \exp^{\Delta_c}_{(k)}(h\,\overline{T}) = \lambda\bar{x}.\ \exp^{\Delta_1,\Delta}_{(k)}(U) = \lambda\bar{x}.\ \exp^{\Delta_1,\Delta,r=_d\lambda\bar{z}.U}_{(k)}(U) = \lambda\bar{x}.\ \exp^{\Delta_1,\Delta,r=_d\lambda\bar{z}.\,U}_{(k)}(r\,\bar{z}) = \exp^{\Delta_1,\Delta,r=_d\lambda\bar{z}.\,U}_{(k)}(\lambda\bar{x}.\, r\,\bar{z})$. □

Corollary 3.2. *If $\Delta_c \rhd \Delta$, then every equation in Δ_c corresponds to an equation in Δ with an equal definitional denotation, and every recursive definition in Δ_c corresponds to a recursive definition in Δ.*

Proof. Directly by structural induction over $\Delta_c \rhd \Delta$. □

3.3 Term Equality and Unifiers

The core ideas for term equality, substitution, and unifiers for the higher-order case are similar to the first-order case. We will only repeat the most essential definitions.

Informally two terms are equal if they have the same definitional expansion. Formally, M is *equal* to M' in a context Δ (i.e. $M \doteq M'$ holds) if for all k, $\exp^{\Delta}_{(k)}(M) = \exp^{\Delta}_{(k)}(M')$.

There are two kinds of substitutions in the higher-order case, substitutions for ordinary variables and substitutions for unification variables. Due to the pattern restriction, the only substitutions for ordinary variables are simultaneous variable renaming that we have seen, and are written in the notation $[\bar{y}/\bar{x}]M$. Substitutions for unification variables remain a special form of unification contexts, but they are now higher-order.

A *substitution* is a contradiction-free unification context where the left-hand side of each unification equation is a unique unification variable followed by a list of bound variables, which is a superset of the free variables occurring on the right-hand side of that equation. Intuitively, the variables following a unification variable serve as λ-binders for its value (on the right-hand side). All unification equations in a substitution are of the form $H^{\blacksquare}\,\bar{x} \doteq U$ or $H^{\bigcirc}\,\bar{x} \doteq N$, where $FV(U) \subseteq \bar{x}$ and $FV(N) \subseteq \bar{x}$. The equation $H^{\blacksquare}\,\bar{x} \doteq U$ or $H^{\bigcirc}\,\bar{x} \doteq N$ is called a *substitution equation* for $H^{\bigcirc}$ or $H^{\blacksquare}$ in Γ. The intuitive meaning of a substitution equation $H^m\,\bar{x} \doteq M$ is that H^m "stands for" $\lambda\bar{x}.\,M$. Since terms are all written in their η-long-form, U or N on the right-hand side of the unification equation should not contain top-level λ-bindings. The set of unification variables that occur on the left-hand sides of the unification equations in a substitution Γ is called the *domain* of the substitution and is denoted $\mathrm{dom}(\Gamma)$. Two substitutions Γ and Γ' are equal if $\mathrm{dom}(\Gamma) = \mathrm{dom}(\Gamma')$, and for all $H^m \in \mathrm{dom}(\Gamma)$, $\exp^{\Gamma}_{(k)}((\lambda\bar{x}.\,H^m\,\bar{x})[\Gamma]) = \exp^{\Gamma'}_{(k)}((\lambda\bar{x}.\,H^m\bar{x})[\Gamma])$, for all k, where $\lambda\bar{x}.\,H^m\,\bar{x}$ is the η-long-form of H^m according to its simple type.

As an example, the Γ, Γ' and Γ'' below are all equal substitutions with the domain $\{S^{\bigcirc}\}$. The main differences are highlighted in blue.

Γ	Γ'	Γ''
$\Gamma = \{$	$\Gamma' = \{$	$\Gamma'' = \{$
$S^{\bigcirc}\,z\,w \doteq r_3\ w,$	$S^{\bigcirc}\,y\,u \doteq r_3\ u,$	$S^{\bigcirc}\,w\,z \doteq r_1\ z,$
$odd =_d get\,(\lambda x.\ even),$	$odd =_d get\,(\lambda x.\ even),$	$odd =_d get\,(\lambda x.\ even),$
$even =_d get\,(\lambda x.\, r_3\,x),$	$even =_d get\,(\lambda x.\, r_3\,x),$	$even =_d get\,(\lambda x.\, r_1\,x),$
$r_3 =_d \lambda x.\ put\,(r_4\ x)\ odd,$	$r_3 =_d \lambda x.\ put\,(r_4\ x)\ odd,$	$r_1 =_d \lambda x.\ put\,(r_5\,x)\ odd,$
$r_4 =_d \lambda x.\,x$	$r_4 =_d \lambda x.\,x$	$r_5 =_d \lambda x.\,x$
$\}$	$\}$	$\}$

We write $U[\Gamma]$ and $N[\Gamma]$ for applying the substitution (for unification variables) to terms. They are defined as follows.

$$(\lambda \bar{x}.h\, N_1\, \ldots\, N_n)[\Gamma] = \lambda \bar{x}.\, h\,(N_1[\Gamma])\, \ldots\, (N_n[\Gamma])$$

$$(\lambda \bar{x}.\, H^{\blacksquare}\, \bar{y})[\Gamma] = \begin{cases} \lambda \bar{x}.\, [\bar{y}/\bar{z}]U'. & \text{if } H^{\blacksquare}\, \bar{z} \doteq U' \in \mathsf{eqs}(\Gamma) \\ \lambda \bar{x}.\, H^{\blacksquare}\, \bar{y} & \text{otherwise} \end{cases}$$

$$(\lambda \bar{x}.\, r\, \bar{y})[\Gamma] = \lambda \bar{x}.\, r\, \bar{y}$$

$$(\lambda \bar{x}.\, H^{\bigcirc}\, \bar{y})[\Gamma] = \begin{cases} \lambda \bar{x}.\, [\bar{y}/\bar{z}]N' & \text{if } H^{\bigcirc}\, \bar{z} \doteq N' \in \mathsf{eqs}(\Gamma) \\ \lambda \bar{x}.\, H^{\bigcirc}\, \bar{y} & \text{otherwise} \end{cases}$$

It is worth noting that the substitution commutes with λ-abstractions that $(\lambda \bar{x}.\, M)[\Gamma] = \lambda \bar{x}.\, (M[\Gamma])$. Substitution also commutes with simultaneous variable renaming that $([\bar{y}/\bar{x}]M)[\Gamma] = [\bar{y}/\bar{x}](M[\Gamma])$. We can show both claims by induction on the structure of M, and the intuition is that substitutions are "closed" substitutions for unification variables.

The application of a substitution Γ to a unification context Δ, the application of a substitution Γ_2 to another substitution Γ_1, and the composition of substitutions $\Gamma_1 \circ \Gamma_2$ (apply Γ_1 and then Γ_2) are analogous counterparts of their first-order definitions.

$$\Delta[\Gamma] \quad = \mathsf{defs}(\Gamma), \{M[\Gamma] \doteq M'[\Gamma] \mid M \doteq M' \in \mathsf{eqs}(\Delta)\}, \{r =_d U[\Gamma] \mid r =_d U \in \mathsf{defs}(\Delta)\}$$

$$\Gamma_1[\Gamma_2] \quad = \mathsf{defs}(\Gamma_2), \{H^m\, \bar{x} \doteq M'[\Gamma_2] \mid H^m\, \bar{x} \doteq M' \in \mathsf{eqs}(\Gamma_1)\}, \{r =_d U[\Gamma_2] \mid r =_d U \in \mathsf{defs}(\Gamma_1)\}$$

$$\Gamma_1 \circ \Gamma_2 \quad = (\Gamma_1[\Gamma_2]), \{H^m\, \bar{x} \doteq M \mid H^m\, \bar{x} \doteq M \in \mathsf{eqs}(\Gamma_2) \wedge H^m \notin \mathsf{dom}(\Gamma_1)\}$$

We repeat the definition of unifiers and the most general unifier for the higher-order case. A *unifier* for a contradiction-free unification context Δ is a substitution Γ such that $UV(\Delta) = \mathsf{dom}(\Gamma)$, and every equation in $\Delta[\Gamma]$ holds. A unification context Δ with $\mathsf{contra} \in \Delta$ has no unifiers. A unifier Γ_1 is *more general* than another unifier Γ_2 if there is a substitution Γ' such that $\Gamma_1 \circ \Gamma' = \Gamma_2$.

As an example, given Γ and Δ defined below, Γ is a unifier of Δ, because every equation in $\Delta[\Gamma]$ holds. The main changes are highlighted in blue. Notice that when carrying out the substitution, the duplicate recursion constants in Γ are renamed by adding a prime (′) sign.

$\Gamma = \{$	$\Delta = \{$	$\Delta[\Gamma] = \{$
$S^{\bigcirc}\, z\, w \doteq r_3\; w,$	$r_1 \doteq odd,$	$r_1 \doteq odd,$
	$r_1 =_d get\,(\lambda x.\, r_2\, x),$	$r_1 =_d get\,(\lambda x.\, r_2\, x),$
	$r_2 =_d \lambda x.\; get\,(\lambda y.\, S^{\bigcirc}\, x\, y),$	$r_2 =_d \lambda x.\; get\,(\lambda y.\, r_3{}'\;\, y),$
	$odd =_d get\,(\lambda x.\; even),$	$odd =_d get\,(\lambda x.\; even),$
	$even =_d get\,(\lambda x.\; r_3\; x),$	$even =_d get\,(\lambda x.\; r_3\; x),$
	$r_3 =_d \lambda x.\; put\,(r_4\; x)\;\, odd,$	$r_3 =_d \lambda x.\; put\,(r_4\; x)\;\, odd,$
	$r_4 =_d \lambda x.\, x$	$r_4 =_d \lambda x.\, x$
$odd =_d get\,(\lambda x.\; even),$	$\}$	$odd' =_d get\,(\lambda x.\, even'),$
$even =_d get\,(\lambda x.\, r_3\, x),$		$even' =_d get\,(\lambda x.\, r_3{}'\, x),$
$r_3 =_d \lambda x.\; put\,(r_4\, x)\;\, odd,$		$r_3{}' =_d \lambda x.\; put\,(r_4{}'\; x)\; odd',$
$r_4 =_d \lambda x.\, x$		$r_4{}' =_d \lambda x.\, x$
$\}$		$\}$

3.4 The Algorithm

We now present the saturation-based algorithm for higher-order rational terms.

In the first-order case, a unification variable is resolved if there is an equation between the unification variable and either a recursion constant or a term with a constant head. In the higher-order case, we have to consider the binding structure, the resolution must only have free variables that appear in the arguments to the unification variable. Also, a unification variable may be resolved by another unification variable that has strictly fewer arguments. A contractive unification variable $H^{\blacksquare}$ is *resolved* if (a) there exists a unification equation $H^{\blacksquare}\, \bar{y} \doteq h\, \bar{z}$ (or its converse) and $\bar{z} \subseteq \bar{y}$, or (b) there exists a unification equation $H^{\blacksquare}\ \bar{y} \doteq G^{\blacksquare}\ \bar{w}$ (or its converse) with $\bar{w} \subsetneq \bar{y}$. [2] $H^{\blacksquare}$ is unresolved otherwise, and is denoted by the judgment $H^{\blacksquare}$ unresolved. Similarly, a contractive unification variable H° is *resolved* if there exists an equation $H^{\circ}\ \bar{y} \doteq r\ \bar{z}$ (or its converse) and $\bar{z} \subseteq \bar{y}$, or there exists an equation $H^{\circ}\ \bar{y} \doteq G^{\circ}\ \bar{w}$ (or its converse) with $\bar{w} \subsetneq \bar{y}$. H° is unresolved otherwise, and is denoted by the judgment H° unresolved.

We saturate the unification context Δ using the rules defined in Figure 1. The saturation rules preserve the definition of all recursion constants. Once saturated, a unifier can be constructed easily. The presence of the constant contra in a unification context indicates that the unification context has no unifiers.

We use the concept of a parameter to ensure the termination of the saturation-based algorithm. The parameters are indicated by bracketed existential quantifiers $(\exists X)$ where X is a parameter that stands for a variable, a recursion constant, a unification variable, or a list of those. The new equation under the existential quantification subsumes any instantiation of the equation [McLaughlin and Pfenning 2009]. When new variables, recursion constants, or unification variables are introduced by one of the rules, the existential quantification ensures that the rule applies (thus the conclusion equations are created) only if there does not exist any instantiation of the conclusion equations in the unification context.

We extend the notation of using overlines to denote lists of unification variables (possibly with arguments) and operations on them. A list of unification variables is written $\overline{G^{\circ}}$ or $\overline{G^{\blacksquare}}$. Also, we write $\overline{G^{\circ}\ \overline{x}}$ to denote the list of applications where each unification variable to applied to $\overline{x}$. For example, $c\ \overline{G^{\circ}\ \overline{x}}$ denotes $c\ (G_1^{\circ}\ x_1\ \ldots\ x_m)\ \ldots\ (G_n^{\circ}\ x_1\ \ldots\ x_m)$. $h\ \overline{\eta\exp(\overline{G^{\circ}\ \overline{x}})}$ denotes a term with head h whose arguments are the result of applying top-level η-expansions to all terms in $\overline{G^{\circ}\ \overline{x}}$ according to the simple type of h. For example, $c\ \eta\exp(\overline{G^{\circ}\ \overline{x}})$ [3] denotes a term of the form $c\ (\lambda\overline{w}_1.\ G_1^{\circ}\ \overline{x}\ \overline{w}_1)\ \ldots\ (\lambda\overline{w}_n.\ G_n^{\circ}\ \overline{x}\ \overline{w}_n)$, i.e.,

$$c\ (\lambda w_{1,1}.\ldots\lambda w_{1,l_1}.\ G_1^{\circ}\ x_1\ \ldots\ x_m\ w_{1,1}\ \ldots\ w_{1,l_1})\ \ldots\ (\lambda w_{n,1}.\ldots\lambda w_{n,l_n}.\ G_n^{\circ}\ x_1\ \ldots\ x_m\ w_{n,1}\ \ldots\ w_{n,l_n})$$

Rule (1)(2) instantiates λ-abstractions, but only when there are no instantiations that are currently present in the context. Rule (8) is called imitation by Huet [1975] because $H^{\blacksquare}$ mimics the behavior of the term $c\ \overline{N}$ on the right. Rules (9)(7) are called projections because $H^{\blacksquare}$ projects its ith argument its head. Rule (10) prunes the variables that are not in common from both H° and r. Rules (11)(12) also serve a similar effect of pruning by removing the extra variables from the arguments to unification variables that cannot be used. Rule (13) unfolds recursive definitions and compares them for equality. Rules (14)(15) ensure that any two resolutions of a unification variable are consistent.

We give an example of how the algorithm operates on the stream processor unification context, now denoted Δ_7 (equations (1) − (7)). At each step, we show some additional equations and the ways they are obtained. We omit the final uninteresting steps when only symmetry and transitivity rules can be applied.

(1) $r_1 \doteq odd$ given
(2) $r_1 =_d get\ (\lambda x.\ r_2\ x)$
(3) $r_2 =_d \lambda x.\ get\ (\lambda y.\ S^{\circ}\ x\ y)$

[2] $\bar{w} \subsetneq \bar{y}$ means $\bar{w}$ is a proper subset of $\bar{y}$. For example, $H^{\blacksquare}\ y \doteq G^{\blacksquare}\ y$ is not a resolution equation, but $H^{\blacksquare}\ y \doteq G^{\blacksquare}$ is, while either equation may appear as a substitution equation for $H^{\blacksquare}$ in a unifier.

[3] We should remark that the non-η-expanded version $c\ \overline{G^{\circ}\ \overline{x}}$ should not appear in the unification context, since we write everything in the η-long form (except the arguments to recursion constants and unification variables). Thus, $h\ \overline{\eta\exp(\overline{G^{\circ}\ \overline{x}})}$ always appears in a conclusion where $\overline{G^{\circ}}$ is fresh (bound by the existential quantifier $(\exists\overline{G^{\circ}})$).

(4) $odd =_d get\ (\lambda x.\ even)$	
(5) $even =_d get\ (\lambda x.\ r_3\ x)$	
(6) $r_3 =_d \lambda x.\ put\ (r_4\ x)\ odd$	
(7) $r_4 =_d \lambda x. x$	
(8) $get\ (\lambda x.\ r_2\ x) \doteq get\ (\lambda x.\ even)$	by Rule (13) on (1), (2) and (4)
(9) $\lambda x.\ r_2\ x \doteq \lambda x.\ even$	by Rule (6) on (8)
(10) $r_2\ z \doteq even$	by Rule (2) on (9), and we verify that there does not exist any equation $(\exists x)\ r_2\ x \doteq even$ in the context Δ_2.
(11) $get\ (\lambda y. S^{\bigcirc}\ z\ y) \doteq get\ (\lambda x.\ r_3\ x)$	by Rule (13) on (10), (3) and (5)
(12) $\lambda y. S^{\bigcirc}\ z\ y \doteq \lambda x.\ r_3\ x$	by Rule (6) on (11)
(13) $S^{\bigcirc}\ z\ w \doteq r_3\ w$	by Rule (2) on (12), and we verify that there does not exist any equation $(\exists x)\ S^{\bigcirc}\ z\ x \doteq r_3\ x$ in the context Δ_2.
(14) ...	by Rule (16)(17)(18)(19)...

The above example used only the structural rules and expansion rules. More examples will be given in Section 3.6.

3.5 Saturated Unification Contexts

We now describe how a substitution $\Gamma = \mathsf{unif}(\Delta)$ can be constructed from a contradiction-free unification context Δ. We will later show that if Δ is a saturated unification context, then $\mathsf{unif}(\Delta)$ is the most general unifier for Δ.

Given any unification context, every unification variable $H^{\blacksquare}$ or $H^{\bigcirc}$ is either resolved or unresolved. Suppose $H^{\blacksquare}$ or $H^{\bigcirc}$ is resolved.

(1) If $H^{\blacksquare}$ is resolved, then there exists an equation $H^{\blacksquare}\ \bar{y} \doteq h\ \bar{z}$ or $H^{\blacksquare}\ \bar{y} \doteq G^{\blacksquare}\ \bar{w}$, with $FV(h\ \bar{z}) \subseteq \bar{y}$ and $\bar{w} \subsetneq \bar{y}$. The equation $H^{\blacksquare}\ \bar{y} \doteq h\ \bar{z}$ or $H^{\blacksquare}\ \bar{y} \doteq G^{\blacksquare}\ \bar{w}$ is called a *resolution equation* for $H^{\blacksquare}$.
(2) If $H^{\bigcirc}$ is resolved, then there exists an equation $H^{\bigcirc}\ \bar{y} \doteq r\ \bar{z}$ or $H^{\bigcirc}\ \bar{y} \doteq G^{\bigcirc}\ \bar{w}$, with $FV(r\ \bar{z}) \subseteq \bar{y}$ and $\bar{w} \subsetneq \bar{y}$. The equation $H^{\bigcirc}\ \bar{y} \doteq r\ \bar{z}$ or $H^{\bigcirc}\ \bar{y} \doteq G^{\bigcirc}\ \bar{w}$ is called a *resolution equation* for $H^{\bigcirc}$.

There may be multiple such equations, but we consistently pick a resolution equation (called *the* resolution equation, or simply the *resolution*[4]) for each unification variable.

(1) Further, the operation of *replacing occurrences of a (resolved) contractive unification variable $H^{\blacksquare}$ by its resolution in a term M* is defined to be M with occurrences of $H^{\blacksquare}\ \bar{x}$ replaced by $[\bar{x}/\bar{y}](h\ \bar{z})$ or $[\bar{x}/\bar{y}](G^{\blacksquare}\ \bar{w})$ according to the resolution equation $H^{\blacksquare}\ \bar{y} \doteq h\ \bar{z}$ or $H^{\blacksquare}\ \bar{y} \doteq G^{\blacksquare}\ \bar{w}$.
(2) Similarly, the operation of *replacing occurrences of a (resolved) recursive unification variable $H^{\bigcirc}$ by its resolution in a term M* is defined to be M with occurrences of $H^{\bigcirc}\ \bar{x}$ replaced by $[\bar{x}/\bar{y}](r\ \bar{z})$ or $[\bar{x}/\bar{y}](G^{\bigcirc}\ \bar{w})$ according to the resolution equation $H^{\bigcirc}\ \bar{y} \doteq r\ \bar{z}$ or $H^{\bigcirc}\ \bar{y} \doteq G^{\bigcirc}\ \bar{w}$.

Now suppose H^m ($m \in \{\blacksquare, \bigcirc\}$) is unresolved. Unresolved unification variables form equivalence classes related by $\doteq$. We pick a *representative unification variable* for each equivalence class, such that if G^m is the representative unification variable for the equivalence class of H^m, there is an equation $H^m\ \bar{y} \doteq G^m\ \bar{z}$, where $\bar{z}$ is a permutation of $\bar{y}$. This equation is called the *representative equation* for H^m. We pick the representative equation in such a way that the right-hand sides $G^m\ \bar{z}$ of all representative equations are equal for all unresolved unification variables in the same

[4]In the higher-order case, resolutions for unification variables are in the form of equations, whereas in the first-order case, the resolutions are terms.

Structural Rules

$$\frac{\lambda\bar{x}.\,U \doteq \lambda\bar{x}.\,U'}{(\exists\bar{x})\;U \doteq U'}(1) \qquad \frac{\lambda\bar{x}.\,N \doteq \lambda\bar{x}.\,N'}{(\exists\bar{x})\;N \doteq N'}(2) \qquad \frac{x\,\overline{N} \doteq c\,\overline{N'}}{\text{contra}}(3) \qquad \frac{x\,\overline{N} \doteq y\,\overline{N'}}{\text{contra}}(x \neq y)(4)$$

$$\frac{c\,\overline{N} \doteq d\,\overline{N'}}{\text{contra}}(c \neq d)(5) \qquad \frac{h\,N_1\,\ldots N_n \doteq h\,N_1'\,\ldots N_n'}{N_1 \doteq N_1', \ldots, N_n \doteq N_n'}(6) \qquad \frac{H^{\blacksquare}\,\bar{y} \doteq z_i\,\overline{N} \qquad z_i \notin \bar{y}}{\text{contra}}(\text{Projection})(7)$$

Resolution Rules

$$\frac{H^{\blacksquare}\,\bar{y} \doteq c\,\overline{N} \qquad H^{\blacksquare}\ \text{unresolved}}{(\exists\overline{G^{\circ}})\;H^{\blacksquare}\,\bar{y} \doteq c\;\eta\exp(\overline{\overline{G^{\circ}\,\bar{y}}})}(\text{Imitation})(8)$$

$$\frac{H^{\blacksquare}\,\bar{y} \doteq y_i\,\overline{N} \qquad y_i \in \bar{y} \qquad H^{\blacksquare}\ \text{unresolved}}{(\exists\overline{G^{\circ}})\;H^{\blacksquare}\,\bar{y} \doteq y_i\;\eta\exp(\overline{\overline{G^{\circ}\,\bar{y}}})}(\text{Projection})(9)$$

$$\frac{H^{\circ}\,\bar{y} \doteq r\,\bar{x} \qquad \bar{x} \not\subseteq \bar{y} \qquad \bar{w} = \bar{x} \cap \bar{y} \qquad H^{\circ}\ \text{unresolved}}{(\exists s, t)(\exists G^{\blacksquare})\;H^{\circ}\,\bar{y} \doteq t\,\bar{w}, r\,\bar{x} \doteq t\,\bar{w}, t =_d \lambda\bar{w}.G^{\blacksquare}\,\bar{w}}(10)$$

$$\frac{\begin{matrix} G^m\,\bar{x} \doteq H^m\,\bar{y} \\ G^m \neq H^m \qquad \bar{z} = \bar{x} \cap \bar{y} \qquad \bar{x} \not\subseteq \bar{y} \wedge \bar{y} \not\subseteq \bar{x} \qquad H^m\ \text{unresolved} \vee G^m\ \text{unresolved} \end{matrix}}{(\exists F^m)\;G^m\,\bar{x} \doteq F^m\,\bar{z}, H^m\,\bar{y} \doteq F^m\,\bar{z}}(m \in \{\blacksquare, \circ\})(11)$$

$$\frac{\begin{matrix} G^m\,\bar{x} \doteq G^m\,\bar{y} \\ \bar{x} = x_1\,\ldots\,x_n \qquad \bar{y} = y_1\,\ldots\,y_n \qquad \bar{z} = \cup_i\{x_i \mid x_i = y_i\} \qquad \bar{x} \neq \bar{y} \qquad G^m\ \text{unresolved} \end{matrix}}{(\exists F^m)\;G^m\,\bar{x} \doteq F^m\,\bar{z}, G^m\,\bar{y} \doteq F^m\,\bar{z}}(m \in \{\blacksquare, \circ\})(12)$$

Expansion, Consistency, Symmetry, and Transitivity

$$\frac{r\,\bar{x} \doteq s\,\bar{y} \qquad r =_d \lambda\bar{z}.\,U_1 \qquad s =_d \lambda\bar{w}.\,U_2}{[\bar{x}/\bar{z}]U_1 \doteq [\bar{y}/\bar{w}]U_2}(13)$$

$$\frac{H^{\blacksquare}\,\bar{x} \doteq U_1 \qquad H^{\blacksquare}\,\bar{y} \doteq U_2 \qquad FV(U_1) \subseteq \bar{x} \qquad FV(U_2) \subseteq \bar{y}}{U_1 \doteq [\bar{x}/\bar{y}]U_2}(14)$$

$$\frac{H^{\circ}\,\bar{x} \doteq N_1 \qquad H^{\circ}\,\bar{y} \doteq N_2 \qquad FV(N_1) \subseteq \bar{x} \qquad FV(N_2) \subseteq \bar{y}}{N_1 \doteq [\bar{x}/\bar{y}]N_2}(15) \qquad \frac{U \doteq U'}{U' \doteq U}(16)$$

$$\frac{N \doteq N'}{N' \doteq N}(17) \qquad \frac{U_1 \doteq U_2 \qquad U_2 \doteq U_3}{U_1 \doteq U_3}(18) \qquad \frac{N_1 \doteq N_2 \qquad N_2 \doteq N_3}{N_1 \doteq N_3}(19)$$

Fig. 1. Unification Rules

equivalence class, i.e. for any other unification variable F^m in the same equivalence class as H^m, $F^m\,\bar{w} \doteq G^m\,\bar{z}$ is picked (and $F^m\,\bar{w}' \doteq G^m\,\bar{z}'$ with $\bar{z}' \neq \bar{z}$ is not picked). The operation of *replacing occurrences of a (unresolved) unification variable H^m by its representative unification variable G^m in a term M* is defined to be M with occurrences of $H^m\,\bar{x}$ replaced by $[\bar{x}/\bar{y}](G^m\,\bar{z})$.

We construct the substitution $\Gamma = \mathsf{unif}(\Delta)$ for a contradiction-free context Δ as follows.

(1) Start with Γ containing all recursive definitions of Δ.
(2) For each resolved unification variable in $UV(\Delta)$, add to Γ the resolution equation of that unification variable.
(3) For each unresolved unification variable in $UV(\Delta)$, add to Γ the representative equation of that unification variable.
(4) Replace the occurrences of resolved unification variables in the right-hand sides and recursive definitions of Γ with their resolutions, and replace the occurrences of unresolved unification variables in the right-hand sides and recursive definitions of Γ with their representatives. Repeat this step until all unification variables the right-hand sides and recursive definitions are representative unification variables for some equivalence class of unresolved unification variables.

As an example, we show how the unifier for Δ_{13} (equations (1) – (13) defined above in Section 3.4), $\Gamma = \mathsf{unif}(\Delta_{13})$ can be constructed. Major changes in each step are highlighted in blue.

(1) Initialize Γ_1 to all recursive definitions of Δ_{13}.
(2) Since H° is resolved, we add its resolution equation to get Γ_2.
(3) There are no unresolved unification variables, we skip step (3).
(4) Replace occurrences of resolved unification variables with their resolutions to get Γ_3.

$\Gamma_1 = \{$	$\Gamma_2 = \{$	$\Gamma_3 = \{$
$r_1 =_d get\,(\lambda x.\,r_2\,x),$	$S^{\circ}\,z\,w \doteq r_3\ w,$	$S^{\circ}\,z\,w \doteq r_3\ w,$
$r_2 =_d \lambda x.\ get\,(\lambda y.\,S^{\circ}\,x\,y),$	$r_1 =_d get\,(\lambda x.\,r_2\,x),$	$r_1 =_d get\,(\lambda x.\,r_2\,x),$
$odd =_d get\,(\lambda x.\ even),$	$r_2 =_d \lambda x.\ get\,(\lambda y.\,S^{\circ}\,x\,y),$	$r_2 =_d \lambda x.\ get\,(\lambda y.\,r_3\ y),$
$even =_d get\,(\lambda x.\,r_3\,x),$	$odd =_d get\,(\lambda x.\ even),$	$odd =_d get\,(\lambda x.\ even),$
$r_3 =_d \lambda x.\ put\,(r_4\ x)\ odd,$	$even =_d get\,(\lambda x.\,r_3\,x),$	$even =_d get\,(\lambda x.\,r_3\,x),$
$r_4 =_d \lambda x.\,x$	$r_3 =_d \lambda x.\ put\,(r_4\ x)\ odd,$	$r_3 =_d \lambda x.\ put\,(r_4\ x)\ odd,$
$\}$	$r_4 =_d \lambda x.\,x$	$r_4 =_d \lambda x.\,x$
	$\}$	$\}$

(5) Note that we may remove unused recursive definitions (r_1, r_2) to get an equivalent substitution Γ_4.
$\Gamma_4 = \{S^{\circ}\,z\,w \doteq r_3\ w, odd =_d get\,(\lambda x.\ even),$
$even =_d get\,(\lambda x.\,r_3\,x), r_3 =_d \lambda x.\ put\,(r_4\ x)\ odd, r_4 =_d \lambda x.\,x\}$

Note that the final substitution Γ_4 is equivalent to the following substitution Γ_c, written in the concrete syntax without flattened definitions. We can easily check that this is a unifier for the concrete unification context Δ_c in Section 3.1.

$\Gamma_c = \{S^{\circ}\,z\,w \doteq r_3\ w, odd =_d get\,(\lambda x.\ get\,(\lambda y.\,r_3\,y)), r_3 =_d \lambda w.\ put\ w\ odd, \}$

3.6 Additional Examples

Recall the signature for stream processors in Section 3.1.

```
sp : cotype.
element : type.
get: (element -> sp) -> sp.
```

```
put: element -> sp -> sp.

odd : sp = get ([x] even).
even : sp = get ([x] put x odd).
```

We have seen an example of how unification can figure out the behavior of the stream processor S after reading two elements of the input, if it behaves the same as odd.

```
?- get ([x] get ([y] S x y)) = odd.
```

$$\Delta_c = \{get\,(\lambda x.\ get\,(\lambda y. S\,x\,y)) \doteq odd, odd =_d get\,(\lambda x.\ even), even =_d get\,(\lambda x.\ put\ x\ odd)\}$$

In the above example, S may depend on both numbers x and y that read from the input stream. We may restrict x to only use the number at index 0, by omitting y from the argument of S.

```
?- get ([x] get ([y] S x)) = odd.
```

$$\Delta_c = \{get\,(\lambda x.\ get\,(\lambda y. S\,x)) \doteq odd, odd =_d get\,(\lambda x.\ even), even =_d get\,(\lambda x.\ put\ x\ odd)\}$$

The odd stream processor outputs an element at index 1, but S doesn't have access to y. This unification problem does not have a solution, and the algorithm eventually adds the constant contra to the unification context. The first six steps are similar to the previous example, but now the equation (13) is no longer a resolution equation for $S^{\bigcirc}$. Note that $\Delta_c \rhd \Delta_7$ (equations (1) − (7)).

(1) $r_1 \doteq odd$	given
(2) $r_1 =_d get\,(\lambda x. r_2\,x)$	
(3) $r_2 =_d \lambda x.\ get\,(\lambda y. S^{\bigcirc}\,x)$	
(4) $odd =_d get\,(\lambda x.\ even)$	
(5) $even =_d get\,(\lambda x.\ r_3\ x)$	
(6) $r_3 =_d \lambda x.\ put\,(r_4\ x)\ odd$	
(7) $r_4 =_d \lambda x. x$	
(8) $get\,(\lambda x.\ r_2\ x) \doteq get\,(\lambda x.\ even)$	by Rule (13) on (1), (2) and (4)
(9) $\lambda x.\ r_2\ x \doteq \lambda x.\ even$	by Rule (6) on (8)
(10) $r_2\ z \doteq even$	by Rule (2) on (9), and we verify that there does not exist any equation $(\exists x)\ r_2\,x \doteq even$ in the context Δ_2.
(11) $get\,(\lambda y. S^{\bigcirc}\,z) \doteq get\,(\lambda x.\ r_3\ x)$	by Rule (13) on (10), (3) and (5)
(12) $\lambda y. S^{\bigcirc}\,z \doteq \lambda x.\ r_3\ x$	by Rule (6) on (11)
(13) $S^{\bigcirc}\,z \doteq r_3\ w$	by Rule (2) on (12), and we verify that there does not exist any equation $(\exists x)\ S^{\bigcirc}\,z \doteq r_3\ x$ in the context Δ_2.
(14) $S^{\bigcirc}\,z \doteq t$	by Rule (10) on (13)
(15) $r_3\ w \doteq t$	
(16) $t =_d G^{\blacksquare}$	
(17) $put\,(r_4\ w)\ odd \doteq G^{\blacksquare}$	by Rule (13) on (15), (6), and (16)
(18) $G^{\blacksquare} \doteq put\,(r_4\ w)\ odd$	by Rule (16) on (17)
(19) $G^{\blacksquare} \doteq put\,F^{\bigcirc}H^{\bigcirc}$	by Rule (8) on (17)
(20) $put\,(r_4\ w)\ odd \doteq put\,F^{\bigcirc}H^{\bigcirc}$	by Rule (18) on (17) and (19)
(21) $r_4\ w \doteq F^{\bigcirc}$	by Rule (6) on (20)
(22) $odd \doteq H^{\bigcirc}$	
(23) $F^{\bigcirc} \doteq r_4\ w$	by Rule (17) on (21)
(24) $F^{\bigcirc} \doteq s$	by Rule (10) on (23)
(25) $r_4\ w \doteq s$	
(26) $s =_d F^{\blacksquare}$	
(27) $w \doteq F^{\blacksquare}$	by Rule (13) on (25), (7), and (26)
(28) $F^{\blacksquare} \doteq w$	by Rule (17) on (27)

(29) contra by Rule (7) on (28)

We now consider some more problems that do not involve *odd* or *even*. For example, we may ask, what is a stream H that outputs the given element and continues as itself.

```
?- [x] put x (H x) = [x] H x.
```

$$\Delta_c = \{\lambda x.\ put\ x\ (H\, x) \doteq \lambda x. H\, x\}$$

The unification algorithm correctly finds a recursive definition for $H^{\bigcirc}$, as seen below, with $\Delta_c \rhd \Delta_3$ (equations (1) – (3)).

(1)$\lambda x.\ r_1\ x \doteq \lambda x. H^{\bigcirc}\ x$ given
(2) $r_1 =_d \lambda x.\ put\ (r_2\ x)\ (H^{\bigcirc}\ x)$
(3) $r_2 =_d \lambda x. x$
(4) $r_1\ z \doteq H^{\bigcirc} z$ by Rule (2) on (1)
(5) . . . . . .

$$\mathsf{unif}(\Delta_4) = \{H^{\bigcirc} z \doteq r_1\ z, r_1 =_d \lambda x.\ put\ (r_2\ x)\ (r_1\ x), r_2 =_d \lambda x. x\}$$

Dually, consider a stream processor S that reads an element and continues as itself, with $\Delta_c \rhd \Delta_2$.

```
?- [x] get ([y] S y) = [x] S x.
```

$$\Delta_c = \{\lambda x.\ get\ (\lambda y. S\, y) \doteq \lambda x. S\, x\}$$

(1)$\lambda x.\ r_1\ x \doteq \lambda x. S^{\bigcirc}\ x$ given
(2) $r_1 =_d \lambda x.\ get\ (\lambda y. S^{\bigcirc}\ y)$
(3) $r_1\ z \doteq S^{\bigcirc} z$ by Rule (2) on (1)
(4) . . . . . .

$$\mathsf{unif}(\Delta_3) = \{S^{\bigcirc}\ z \doteq r_1\ z, r_1 =_d \lambda x.\ get\ (\lambda y.\ r_1\ y)\}$$

Here, the definition r_1 never uses its argument. Our unification algorithm will not actively prune the arguments to recursion constants unless triggered by a unification equation like in Rule (10).

Finally, consider the following unification problem.

```
?- get ([x] get ([y] H x)) = get ([x] get ([y] S y)).
```

$$\Delta_c = \{get\ (\lambda x.\ get\ (\lambda y. H\, x)) \doteq get\ (\lambda x.\ get\ (\lambda y. S\, y))\}$$

After reading two input elements, continuation H may use the first element, and continuation S may use the second element, but H and S have to be equal. Unification correctly finds that neither H nor S can use their argument.

(1) $r_1 \doteq r_3$ given
(2) $r_1 =_d get\ (\lambda x. r_2\, x)$
(3) $r_2 =_d \lambda x.\ get\ (\lambda y. H^{\bigcirc}\ x)$
(4) $r_3 =_d get\ (\lambda x. r_4\, x)$
(5) $r_4 =_d \lambda x.\ get\ (\lambda y. S^{\bigcirc}\ y)$
(6) $get\ (\lambda x.\ r_2\ x) \doteq get\ (\lambda x.\ r_4\ x)$ by Rule (13) on (1), (2) and (4)
(7) $\lambda x.\ r_2\ x \doteq \lambda x.\ r_4\ x$ by Rule (6) on (6)
(8) $r_2\ z \doteq r_4\ z$ by Rule (2) on (7)
(9) $get\ (\lambda y. H^{\bigcirc}\ z) \doteq get\ (\lambda y. S^{\bigcirc}\ y)$ by Rule (13) on (8), (3) and (5)
(10) $\lambda y. H^{\bigcirc}\ z \doteq \lambda y. S^{\bigcirc}\ y$ by Rule (6) on (9)
(11) $H^{\bigcirc}\ z \doteq S^{\bigcirc}\ w$ by Rule (2) on (10)
(12) $H^{\bigcirc}\ z \doteq F^{\bigcirc}$ by Rule (11) on (11)
(13) $S^{\bigcirc}\ w \doteq F^{\bigcirc}$
(14) . . . . . .

Now the unifier for Δ_{13} is

$$\mathsf{unif}(\Delta_{13}) = \{H^{\bigcirc}\, z \doteq F^{\bigcirc}, S^{\bigcirc}\, w \doteq F^{\bigcirc}, F^{\bigcirc} \doteq F^{\bigcirc}\}$$

3.7 Correctness of the Algorithm

We prove the termination, soundness, and completeness of the algorithm, by a similar strategy as the first-order case. Any context will saturate in a finite number of steps, and the unifiers are preserved in each step modulo domain restriction, to be defined.

Lemma 3.3 (Preservation of Well-formed Unification Contexts). *The pattern restriction, β-normal-η-long forms, and typing are respected by the unification rules.*

Proof. Directly by analyzing the rules. □

Theorem 3.4 (Correctness of Unifiers). *If Δ is a saturated contradiction-free unification context, and $\Gamma = \mathsf{unif}(\Delta)$ is the most general unifier for Δ.*

Proof. The proof largely follows the structure of the first-order case. We repeat the entire proof here for completeness.

To show Γ is a unifier, we need to show that $\mathsf{dom}(\Gamma) = UV(\Delta)$, which is true by definition, and that every equation in $\Delta[\Gamma]$ holds. It suffices to show the following.

(1) For all $U_1 \doteq U_2$ in Δ, $\mathsf{exp}^{\Delta[\Gamma]}_{(k)}(U_1[\Gamma]) = \mathsf{exp}^{\Delta[\Gamma]}_{(k)}(U_2[\Gamma])$.
(2) For all $N_1 \doteq N_2$ in Δ, $\mathsf{exp}^{\Delta[\Gamma]}_{(k)}(N_1[\Gamma]) = \mathsf{exp}^{\Delta[\Gamma]}_{(k)}(N_2[\Gamma])$.

We show the following two claims simultaneous by lexicographic induction on (k, and the structure of U or N), where claim (2) may refer to claim (1) without decreasing k. Both claims are trivial when $k = 0$. Consider the case when $k > 0$, we show (1) and (2) by case analysis on the structure of $U_1 \doteq U_2$ and $N_1 \doteq N_2$.

We state some facts about the construction of $\mathsf{unif}(\Delta)$. We note that if the substitution equation for $H^{\blacksquare}$ is $H^{\blacksquare}\, \bar{w} \doteq F^{\blacksquare}\, \bar{z}$ (F is necessarily unresolved by step (4) of the construction for $\Gamma = \mathsf{unif}(\Delta)$), then the substitution equation will also appear in Δ due to transitivity and the consistency of resolution rule. Also, if the substitution equation for $H^{\blacksquare}$ is $H^{\blacksquare}\, \bar{w} \doteq h\, N_1\, \ldots\, N_n$, there exists equations (not necessarily picked as resolution equations) $H^{\blacksquare}\, \bar{w} \doteq h\, N_1'\, \ldots\, N_n'$ (with $FV(h\, N_1'\, \ldots\, N_n') \subseteq \bar{w}$), and $N_1 \doteq N_1', \ldots, N_n \doteq N_n'$ in Δ, by inspecting the process of obtaining Γ and the structural rule (6). For recursive unification variables $H^{\bigcirc}$, the substitution equations $H^{\bigcirc}\, \bar{w} \doteq F^{\bigcirc}\, \bar{z}$ and $H^{\bigcirc}\, \bar{w} \doteq r\, \bar{z}$ will appear in Δ due to transitivity and the consistency of resolution (rule (15)).

(a) If U_1 or U_2 contains top-level λ-abstractions, then they must have equal number of λ-abstractions due to typing and η-long forms. Due to the commutation of the definitional expansion and λ-abstractions and the commutation between substitution and λ-abstractions, the result follows by induction hypothesis. The cases (b)(c)(d) below show the claim when U_1 and U_2 do not have top-level λ-abstractions.
(b) Both U_1 and U_2 have constants or variables as their heads. Since $\mathsf{contra} \notin \Delta$, they must have identical constant or variable heads. Now let $U_1 = h\, N_1\, \ldots\, N_n$ and $U_2 = h\, N_1'\, \ldots\, N_n'$. Since Δ is saturated, we have $N_i \doteq N_i'$ for all $1 \leq i \leq n$. The result then follows from the fact that each N_i and N_i' have equal definitional expansion up to depth $k - 1$ by induction hypothesis.
(c) One of U_1 and U_2 is a unification variable, and the other has a constant as its head. Without loss of generality, assume $U_1 = H^{\blacksquare}\, \bar{x}$ and $U_2 = h\, \overline{N}$. We have two cases, either $FV(h\, \overline{N}) \subseteq \bar{x}$ or not.

(i) If $FV(h\,\overline{N}) \subseteq \bar{x}$, then this is a resolution equation. If this is the resolution equation used in Γ, then we're done. Otherwise, rule (14) ensures that there is an equation between U_2 and the resolution equation used in Γ, and the rest follows from the case (b) above.

(ii) If $FV(h\,\overline{N}) \not\subseteq \bar{x}$, In this case either $H^{\blacksquare}$ is resolved or unresolved. $H^{\blacksquare}$ cannot be unresolved, since otherwise rule (8) or (9) would apply. If $H^{\blacksquare}$ is resolved, then $H^{\blacksquare}\,\bar{x} \doteq h\,\overline{N'}$ is a substitution equation, and the result follows from IH on the necessary equations between $\overline{N'}$ and $\overline{N}$.

(d) Both U_1 and U_2 have contractive unification variables as their heads. Let $U_1 = H^{\blacksquare}\,\bar{x}$ and $U_2 = G^{\blacksquare}\,\bar{y}$. Due to saturation, WLOG, there are three cases, both unification variables are unresolved, only one is unresolved, or both are resolved. We consider them one by one.

(i) Both are unresolved. If $H^{\blacksquare}$ is equal to $G^{\blacksquare}$ (they are the same unification variable), then $\bar{x} = \bar{y}$ (position-wise) since otherwise rule (12) will apply, and it would have a resolution. Otherwise, suppose $H^{\blacksquare} \neq G^{\blacksquare}$, Since they are in the same equivalence class, for some representative unification variable be $F^{\blacksquare}$, we have the representative equations $H^{\blacksquare}\,\bar{w} \doteq F^{\blacksquare}\,\bar{z}$ and $G^{\blacksquare}\,\bar{u} \doteq F^{\blacksquare}\,\bar{z}$ in Γ. Here $\bar{w}$ and $\bar{x}$ may differ. By rule (14), on $H^{\blacksquare}$, we have equations $G\,\bar{y} \doteq [\bar{x}/\bar{w}](F^{\blacksquare}\,\bar{z})$ and similarly $H^{\blacksquare}\,\bar{x} \doteq [\bar{x}/\bar{w}](F^{\blacksquare}\,\bar{z})$ in Δ By symmetry and transitivity, we have $[\bar{x}/\bar{w}](F^{\blacksquare}\,\bar{z}) \doteq [\bar{y}/\bar{u}](F^{\blacksquare}\,\bar{z})$ in Δ. We have just shown that $[\bar{x}/\bar{w}](F^{\blacksquare}\,\bar{z})$ and $[\bar{y}/\bar{u}](F^{\blacksquare}\,\bar{z})$ are syntactically equal (otherwise they will be resolved by rule (12)). But now $U_1[\Gamma] = (H^{\blacksquare}\,\bar{x})[\Gamma] = [\bar{x}/\bar{w}](F^{\blacksquare}\,\bar{z}) = [\bar{y}/\bar{u}](F^{\blacksquare}\,\bar{z}) = (G^{\blacksquare}\,\bar{y})[\Gamma] = U_2[\Gamma]$.

(ii) Only one of them is unresolved. WLOG, $H^{\blacksquare}$ is unresolved and $G^{\blacksquare}$ is resolved. We have $G^{\blacksquare}\,\bar{u} \doteq F^{\blacksquare}\,\bar{z}$ or $G^{\blacksquare}\,\bar{u} \doteq h\,\overline{N}$ in Γ. In the first case $F^{\blacksquare}\,\bar{z}$ is unresolved (otherwise it would have been replaced in step (4)), and then $H^{\blacksquare}$ and $F^{\blacksquare}$ are in the same equivalence class, and the rest follows from the case (d)(i) above. In the second case, we would have an equation $H^{\blacksquare}\,\bar{x} \doteq [\bar{y}/\bar{u}](h\,\overline{N})$. But now $H^{\blacksquare}$ could be resolved by rule (8) or (9).

(iii) Both are resolved. Suppose $H^{\blacksquare}\,\bar{x} \doteq U_{H^{\blacksquare}}$ and $G^{\blacksquare}\,\bar{y} \doteq U_{G^{\blacksquare}}$ are substitution equations in Γ. It cannot be the case that only one of $U_{H^{\blacksquare}}$ and $U_{G^{\blacksquare}}$ has unresolved unification variables as the head, and the other has a constant or a variable as the head. Since transitivity and rule (14) ensure an equation between $U_{H^{\blacksquare}}$ and $U_{G^{\blacksquare}}$, and the other would be resolved by rule (8) or (9). Thus, both $U_{H^{\blacksquare}}$ and $U_{G^{\blacksquare}}$ have unresolved unification variables as the head, or both have constant or variables as the head. In the first case, the result follows from the case (d)(i) above. In the second case, let $U_{H^{\blacksquare}} = h\,\overline{N}$ and $U_{G^{\blacksquare}} = h\,\overline{N'}$, there are equations $U_1 \doteq h\,\overline{N''}$ and $U_2 \doteq h\,\overline{N'''}$ in Δ, with equations between $\overline{N}$ and $\overline{N''}$ (pairwise), and similarly for $\overline{N'}$ and $\overline{N'''}$. By transitivity, there is an equation $h\,\overline{N''} \doteq h\,\overline{N'''}$, and thus there are equations between $\overline{N''}$ and $\overline{N'''}$ (pairwise). Then the result follows by IH to show $\overline{N}, \overline{N'}, \overline{N''}, \overline{N'''}$ all have equal definitional expansions up to depth $k-1$.

(e) The case where N_1 or N_2 contains top-level λ-abstractions is similar to the case (a), and we show subsequently the cases when N_1 and N_2 do not contain top-level λ-abstractions.

(f) Both N_1 and N_2 have recursion constants as heads. Let $N_1 = r\,\bar{x}$, where $r =_d \bar{w}.\,U_1 \in \Delta$ and $N_2 = s\,\bar{y}$, where $s =_d \lambda\bar{u}.\,U_2 \in \Delta$. Since Δ is saturated, $[\bar{x}/\bar{w}]U_1 \doteq [\bar{y}/\bar{u}]U_2 \in \Delta$. By IH, $\exp_{(k)}(([\bar{x}/\bar{w}]U_1)[\Gamma]) = \exp_{(k)}(([\bar{y}/\bar{u}]U_2)[\Gamma])$, and then $\exp_{(k)}(N_1[\Gamma]) = \exp_{(k)}((r\,\bar{x})[\Gamma]) = \exp_{(k)}(([\bar{x}/\bar{w}]U_1)[\Gamma]) = \exp_{(k)}(([\bar{y}/\bar{u}]U_2)[\Gamma]) = \exp_{(k)}((s\,\bar{y})[\Gamma]) = \exp_{(k)}(N_2[\Gamma])$.

(g) One of N_1 and N_2 is a unification variable, and the other has a recursion constant as its head. WLOG, assume $N_1 = H^{\circ}\,\bar{x}$ and $N_2 = r\,\bar{y}$. We have two cases, either $FV(r\,\bar{y}) \subseteq \bar{x}$ or not.

(i) If $FV(r\,\bar{y}) \subseteq \bar{x}$, then this is a resolution equation. If this is the substitution equation used in Γ, then we're done. Otherwise, rule (15) ensures that there is an equation between N_2 and the resolution equation used in Γ, and the rest follows from the case (f) above.

(ii) If $FV(r\,\bar{y}) \nsubseteq \bar{x}$, In this case either $H^{\bigcirc}$ is resolved or unresolved. $H^{\bigcirc}$ cannot be unresolved, since otherwise rule (10) would apply. If $H^{\bigcirc}$ is resolved, then let $H^{\bigcirc}\,\bar{x} \doteq s\,\bar{z}$ be a substitution equation. By transitivity, there is an equation between $s\,\bar{z}$ and $r\,\bar{y}$. By the case (f), $\exp_{(k)}(s\,\bar{z}) = \exp_{(k)}(r\,\bar{y})$,

(h) Both N_1 and N_2 have recursive unification variables as their heads. Let $N_1 = H^{\bigcirc}\,\bar{x}$ and $N_2 = G^{\bigcirc}\,\bar{y}$. Due to saturation, WLOG, there are three cases, both unification variables are unresolved, only one is unresolved, or both are resolved. We consider them one by one.

(i) Both are unresolved. This is exactly analogous to the case (d)(i).

(ii) Only one of them is unresolved. This is exactly analogous to the case (d)(ii), except that in the case the resolution is a recursion constant, the unresolved unification variables may be resolved by rule (10).

(iii) Both are resolved. Suppose $H^{\bigcirc}\,\bar{x} \doteq N_{H^{\circ}}$ and $G^{\bigcirc}\,\bar{y} \doteq N_{G^{\circ}}$ are substitution equations in Γ. By a similar reasoning as (d)(iii), both $N_{H^{\circ}}$ and $N_{G^{\circ}}$ have unresolved unification variables as heads, or both have recursion constants as the heads. IN the first case, the equality can be established by (h)(i). In the latter case, there is an equation $N_{H^{\circ}} \doteq N_{G^{\circ}}$ due to transitivity and the rest follows by the case (f).

To show Γ is the most general unifier, given any other unifier Γ_2 of Δ, it suffices to construct a unifier Γ_1 such that $\Gamma \circ \Gamma_1 = \Gamma_2$. But the construction of Γ_1 is easy: Γ_2 must map resolved unification variables analogously as Γ (otherwise a contradiction will arise), and it may choose to map equivalence classes of unresolved unification variables freely. Γ_1 simply records how unresolved unification variables are mapped in Γ'.

□

The preservation of β-normal-η-long forms guarantees that the number of variables following any unification variable is constant throughout. We define the *width* of a unification variable to be the number of variables following it. For example, if $F^{\blacksquare}\,x\,y\,z$ appears in a unification context Δ, then the width of $F^{\blacksquare}$ is 3. Similarly, we define the *width* of a unification constant to be the number of variables following it. A recursion constant r is *pruned* if there exists an equation $r\,\bar{x} = s\,\bar{y}$ such that $\bar{y} \subsetneq \bar{x}$, and r is *unpruned* otherwise.

Theorem 3.5 (Termination). *The algorithm always terminates.*

Proof. We observe that terms in the unification contexts are shallow as defined by the grammar, and all terms are well-typed. Given a bounded amount of variables, unification variables, and recursion constants, there can only be finitely many equations and recursive definitions in a unification context. The rules that create new unification variables, variables, or recursion constants are rules (1)(2)(8)(9)(10)(11)(12), it suffices to show that these rules can only be applied finitely many times.

First we show that given a bounded amount of unification variables and recursion constants, the rules (1)(2) can only be applied finitely many times. Since everything is well-typed, the maximum depth and width for terms are bounded. Then, there are only finitely many equations modulo simultaneous variable renaming, and the subsumption $\exists\bar{x}$ in the conclusion of the rule (1)(2) prevents additional equations from being created that are merely variable renaming of existing equations.

Then, it suffices to show the rules (8)(9)(10)(11)(12) can only be applied finitely many times. We associate with each unification context a lexicographic multi-set order $\langle A, B, C\rangle$, where A, B, C are multisets of natural numbers defined below, and show that each rule that creates new unification variables or recursion constants strictly decreases this order. The multiset order [Dershowitz and Manna 1979] states that a multiset of natural numbers X is considered smaller than another multiset Y if X can be obtained from Y by removing a natural number n and adding a finite number of natural numbers that are strictly smaller than n. The order $\langle A, B, C \rangle$ is given by

(1) $A = \{\mathsf{width}(r) \mid r =_d U \in \Delta, r \text{ unpruned}\}$ is the multiset of widths of all unpruned recursion constants.
(2) $B = \{\mathsf{width}(H^{\blacksquare}) \mid H^{\blacksquare} \in UV(\Delta), H^{\blacksquare} \text{ unresolved}\}$ is the multiset of widths of all unresolved contractive unification variables.
(3) $C = \{\mathsf{width}(H^{\bigcirc}) \mid H^{\bigcirc} \in UV(\Delta), H^{\bigcirc} \text{ unresolved}\}$ is the multiset of widths of all unresolved recursive unification variables.

For example, we could decrease the order $\langle A, B, C \rangle$ by resolving a contractive unification variable, and adding arbitrarily many recursive unification variables of any width.

First, it is easy to see that no rules can ever increase this order except rules (8)(9)(10)(11)(12): once a unification variable is resolved, it remains resolved, and once a recursion constant is pruned, it remains pruned. Both conditions rely on the existence of certain equations and rules never remove equations from the unification context. Then we show each of the rules (8)(9)(10)(11)(12) strictly decreases the order $\langle A, B, C \rangle$.

Rule (8) or (9) removes one unresolved contractive unification variable and adds a finite number of recursive unification variables.

Rule (10) prunes a recursion constant and adds a contractive unification variable.

Each of the rules (11) and (12) resolves a recursive (resp. contractive) unification variable and adds a recursive (resp. contractive) of a smaller width.

□

Lemma 3.6. *Given unification contexts Δ and Δ', $\mathsf{eqs}(\Delta) \subseteq \mathsf{eqs}(\Delta')$, $\mathsf{defs}(\Delta) = \mathsf{defs}(\Delta')$, $UV(\Delta) = UV(\Delta')$, then any unifier of Δ' is a unifier of Δ.*

Proof. The proof is essentially the same as the proof for Lemma 2.5, by observing that definitional expansion does not depend on unification equations but only on recursive definitions. □

Lemma 3.7. *If Γ is a unifier for Δ, then Γ is a unifier for Δ' where Δ' has all recursive definitions of Δ and additional true equations $M \doteq M'$ in the sense that $\mathsf{exp}^{\Delta[\Gamma]}_{(k)}(M[\Gamma]) = \mathsf{exp}^{\Delta[\Gamma]}_{(k)}(M'[\Gamma])$.*

Proof. Because definitional expansion depends only on recursive definitions, we have $\mathsf{exp}^{\Delta[\Gamma]}_{(k)}(M) = \mathsf{exp}^{\Delta'[\Gamma]}_{(k)}(M)$ for all k and M. □

Let $\Gamma'|_{UV(\Delta)}$ be the substitution Γ' with domain restricted to $UV(\Delta)$, i.e., removing all substitution equations of Γ' if the unification variable on the left-hand side does not belong to $UV(\Delta)$. We have the following two lemmas.

Lemma 3.8. *Given unification contexts Δ and Δ', $\mathsf{eqs}(\Delta) \subseteq \mathsf{eqs}(\Delta')$, $\mathsf{defs}(\Delta) \subseteq \mathsf{defs}(\Delta')$, $UV(\Delta) \subseteq UV(\Delta')$, for any unifier Γ of Δ', $\Gamma|_{UV(\Delta)}$ is a unifier of Δ.*

Proof. This proof is similar to the proof of Lemma 2.5. We repeat the entire proof here.

Let Γ be a unifier of Δ', all unification equations of $\Delta'[\Gamma]$ hold. Take any $M \doteq M' \in \Delta$, we know $\mathsf{exp}^{\Delta'[\Gamma]}_{(k)}(M[\Gamma]) = \mathsf{exp}^{\Delta'[\Gamma]}_{(k)}(M'[\Gamma])$, and all recursion constants in M occur in Δ (because unification context has to be well-formed). It suffices to show $\mathsf{exp}^{\Delta[\Gamma]}_{(k)}(M[\Gamma]) = \mathsf{exp}^{\Delta[\Gamma]}_{(k)}(M'[\Gamma])$

by showing $\exp^{\Delta[\Gamma]}_{(k)}(M[\Gamma]) = \exp^{\Delta'[\Gamma]}_{(k)}(M[\Gamma])$. But since definitional expansions only depend on recursive definitions that actually occur in Δ, we have

$$\exp^{\Delta[\Gamma]}_{(k)}(M[\Gamma]) = \exp^{\mathsf{defs}(\Delta[\Gamma])}_{(k)}(M[\Gamma]) = \exp^{\mathsf{defs}(\Delta'[\Gamma])}_{(k)}(M[\Gamma]) = \exp^{\Delta'[\Gamma]}_{(k)}(M[\Gamma])$$

□

Lemma 3.9. *Domain restrictions $-|_{UV(\Delta)}$ preserves the ordering on unifiers.*

Given a substitution $\Gamma_1 \circ \Gamma_2 = \Gamma_3$, let $S \subseteq \mathrm{dom}(\Gamma_1)$, then there exists Γ_2' such that $(\Gamma_1|_S) \circ \Gamma_2' = (\Gamma_3|_S)$. Moreover, $\Gamma_2' = \Gamma_2|_{FUV(\Gamma_1|_S)}$, where the set of free unification variables of a substitution, $FUV(\Gamma)$, is the set of unification variables that occur on the right-hand sides and recursive definitions of the substitution Γ.

Proof. For any substitution equation $H^m\,\bar{x} \doteq M \in \Gamma_1|_S$, the result of applying Γ_2 to M is the same as the result of applying $\Gamma_2|_{FUV(\Gamma_1|_S)}$ to M.

□

Theorem 3.10 (Correspondence). *If Δ transforms into Δ' by applying one of the rules to some equation in Δ, then the set of unifiers of Δ coincides with the set of unifiers in Δ' with domains restricted to $UV(\Delta)$. Moreover, domain restriction preserves most general unifiers.*

Proof. If Δ' contains contra then there is no unifier for Δ', and we can show in each case that there is no unifier for Δ by inspecting rules (3)(4)(5)(7), and the case where contra is already present in Δ. Otherwise, assume contra $\notin \Delta'$.

For the rules that do not add new unification variables. Every unifier for Δ' is also a unifier of Δ by Lemma 3.6. And every unifier of Δ is also a unifier for Δ' by Lemma 3.7. We show rule (13)(14) as examples for applying Lemma 3.7.

Rule (13), given $r\,\bar{x} \doteq s\,\bar{y}$, $r =_d \lambda z.\,U_1$, and $s =_d \lambda w.\,U_2$ in Δ, it adds the equation $[\bar{x}/\bar{z}]U_1 \doteq [\bar{y}/\bar{w}]U_2$ in Δ'. By Lemma 3.7, it suffices to show $\exp^{\Delta[\Gamma]}_{(k)}(([\bar{x}/\bar{z}]U_1)[\Gamma]) = \exp^{\Delta[\Gamma]}_{(k)}(([\bar{y}/\bar{w}]U_2)[\Gamma])$. Since Γ is a unifier for Δ, we have $\exp^{\Delta[\Gamma]}_{(k)}((r\,\bar{x})[\Gamma]) = \exp^{\Delta[\Gamma]}_{(k)}((s\,\bar{y})[\Gamma])$, but $\exp^{\Delta[\Gamma]}_{(k)}((r\,\bar{x})[\Gamma]) = \exp^{\Delta[\Gamma]}_{(k)}(r\,\bar{x}) = \exp^{\Delta[\Gamma]}_{(k)}([\bar{x}/\bar{z}](U_1[\Gamma]))$ and similarly for $(s\,\bar{y})$.

Rule (14), given $H^{\blacksquare}\,\bar{x} \doteq U_1$ and $H^{\blacksquare}\,\bar{y} \doteq U_2$ in Δ, it adds the equation $U_1 \doteq [\bar{x}/\bar{y}]U_2$ to Δ'. Let Γ be a unifier for Δ, we want to show Γ is a unifier for Δ'. By Lemma 3.7, it suffices to show $\exp^{\Delta[\Gamma]}_{(k)}(U_1[\Gamma]) = \exp^{\Delta[\Gamma]}_{(k)}(([\bar{x}/\bar{y}]U_2)[\Gamma])$. Suppose $H^{\blacksquare}\,\bar{w} \doteq U_{H^{\blacksquare}}$ is the substitution equation in Γ, and then $\exp^{\Delta[\Gamma]}_{(k)}(U_1[\Gamma]) = \exp^{\Delta[\Gamma]}_{(k)}((H^{\blacksquare}\,\bar{x})[\Gamma]) = \exp^{\Delta[\Gamma]}_{(k)}([\bar{x}/\bar{w}]U_{H^{\blacksquare}}) = \exp^{\Delta[\Gamma]}_{(k)}([\bar{x}/\bar{y}][\bar{y}/\bar{w}]U_{H^{\blacksquare}}) = [\bar{x}/\bar{y}]\exp^{\Delta[\Gamma]}_{(k)}([\bar{y}/\bar{w}]U_{H^{\blacksquare}}) = [\bar{x}/\bar{y}]\exp^{\Delta[\Gamma]}_{(k)}((H^{\blacksquare}\,\bar{y})[\Gamma]) = [\bar{x}/\bar{y}]\exp^{\Delta[\Gamma]}_{(k)}(U_2[\Gamma]) = \exp^{\Delta[\Gamma]}_{(k)}([\bar{x}/\bar{y}](U_2[\Gamma])) = \exp^{\Delta[\Gamma]}_{(k)}(([\bar{x}/\bar{y}]U_2)[\Gamma])$.

The rest of the cases except rules (8)(9)(10)(11)(12) are similar. It is obvious that the identity mapping (as a trivial domain restriction) preserves the most general unifiers.

We show that for rules (8)(9)(10)(11)(12), if Γ' is a unifier for Δ', and then $\Gamma = \Gamma'|_{UV(\Delta)}$ is the unifier for Δ. Our proof shares the essential ideas of Miller [1991] and Huet [1975]'s proofs.

Rules (8)(9), we have $H^{\blacksquare}\,\bar{y} \doteq h\,N_1\,\ldots N_n \in \Delta$, and

$$H^{\blacksquare}\,\bar{y} \doteq h\,(\lambda\bar{x}_1.\,G_1^{\bigcirc}\,\bar{y}\,\bar{x}_1)\,\ldots\,(\lambda\bar{x}_n.\,G_n^{\bigcirc}\,\bar{y}\,\bar{x}_n) \in \Delta'$$

Given Γ' is a unifier for Δ', it follows from Lemma 3.8 that Γ is a unifier for Δ. To show that the restriction $-|_{UV(\Delta)}$ preserves unifiers, it suffices to show that given any Γ that is a unifier of Δ, there exists a unique Γ' such that $\Gamma'|_{UV(\Delta)}$ is Γ. Given a unifier Γ, the substitution equation for $H^{\blacksquare}$ is $H^{\blacksquare}\,\bar{z} \doteq h\,(\lambda\bar{x}_1.\,N_1')\,\ldots\,(\lambda\bar{x}_n.\,N_n')$. The corresponding Γ' will map $H^{\blacksquare}$ analogously, and will have the substitution equation for $G_i^{\bigcirc}$ as $G_i^{\bigcirc}\,\bar{x}_i\,\bar{z} \doteq N_i'$. Γ' is a unifier for Δ' is by a similar reasoning as the proof for Lemma 3.7. This Γ' is unique because any different mapping of $G_i^{\bigcirc}$ (modulo definitional expansion) will make the additional equation in Δ' false.

Rule (10), we have $H^{\bigcirc}\,\bar{y} \doteq r\,\bar{x} \in \Delta$, and

$$H^{\bigcirc}\,\bar{y} \doteq t\,\bar{w}, r\,\bar{x} \doteq t\,\bar{w}, t =_d \lambda\bar{w}.\,G^{\blacksquare}\,\bar{w} \in \Delta'$$

with $\bar{w} = \bar{x} \cap \bar{y}$. Given Γ' is a unifier for Δ', it follows from Lemma 3.8 that Γ is a unifier for Δ. To show that the restriction $-|_{UV(\Delta)}$ preserves unifiers, it suffices to show that given any Γ that is a unifier of Δ, there exists a unique Γ' such that $\Gamma'|_{UV(\Delta)}$ is Γ. Given a unifier Γ, the substitution equation for $H^{\bigcirc}$ is $H^{\bigcirc}\,\bar{y} \doteq s\,\bar{w}$ with $s =_d \lambda\bar{w}.\,U \in \Gamma$, [5]. $H^{\bigcirc}$ cannot be mapped to another recursive unification variable because of the equation $H^{\bigcirc}\,\bar{y} \doteq r\,\bar{x}$. The corresponding Γ' will map $H^{\bigcirc}$ analogously, and will have the substitution equation for $G^{\blacksquare}$ as $G^{\blacksquare}\,\bar{w} \doteq U$. This Γ' is unique because any different mapping of $G^{\blacksquare}$ (modulo definitional expansion) will make the additional equation in Δ' false.

Rules (11)(12) follows a similar argument. We elide the full development and remark that Γ' will map the additional unification variable F^m analogously as Γ maps G^m.

$-|_{UV(\Delta)}$ preserves the most general unifiers since the substitution that mediates between the most general unifier Γ' and any more specific Γ_2 is a substitution whose restriction mediates between $\Gamma'|_{UV(\Delta)}$ and $\Gamma_2|_{UV(\Delta)}$ by Lemma 3.9.

□

4 RELATED WORK

The unification algorithm for the first-order terms was first developed by Robinson [1965] as a procedure for implementing resolution. Jaffar [1984] gave an efficient unification algorithm for first-order rational trees based on the system of equations presentation [Martelli and Montanari 1982]. Huet [1975] has discovered a pre-unification algorithm for general higher-order terms. Although general higher-order unification is undecidable and does not have most general unifiers [Huet 1973], Miller [1991] discovered that if arguments to unification variables are restricted to pairwise distinct bound variables, decidability and most general unifiers can be recovered. A similar idea of restricting the arguments to bound variables gives a formulation of regular Böhm trees [Huet 1998] with decidable equality. Our use of a signature for representing recursive definitions directly follows that of CoLF [Chen and Pfenning 2023].

Nominal unification is an alternative way of carrying out higher-order unification [Urban 2010; Urban et al. 2004]. It is encodable in higher-order pattern unification and higher-order pattern unification can be encoded in nominal unification. Schmidt-Schauß et al. [2022] have presented a nominal unification algorithm for a version of cyclic λ-calculi by Ariola and Blom [1997]. However, their cyclic λ-calculi has a different criterion for term equality than ours.

5 CONCLUSION

We have presented a saturation-based unification algorithm for finding most general unifiers for higher-order rational terms ($\bot$-free regular Böhm trees). We have shown the termination, soundness, and completeness of this algorithm. The main complexity is to arrange the conditions for applying the rules to ensure termination. We once again find Miller's pattern fragment to be fundamental in determining the most general unifiers in the presence of higher-order terms.

[5] In practice, the substitution equation may be $H^{\bigcirc}\,\bar{z} \doteq s\,\bar{u}$, we can α-rename it to $H^{\bigcirc}\,\bar{y} \doteq s\,([\bar{y}/\bar{z}]\bar{u})$. It might be the case that $([\bar{y}/\bar{z}]\bar{u}) \subsetneq \bar{w}$, with $s =_d \lambda([\bar{y}/\bar{z}]\bar{u}).\,U$, but we can always construct another definition $q =_d \lambda\bar{w}.\,U$ and set $H^{\bigcirc}\,\bar{y} \doteq q\,\bar{w}$.

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